1. RADIATIVE RECOMBINATION RATES AND THE MILNE RELATION

For H, recombination is electron-proton collisions expressed by the reaction

$$e + H^+ \rightarrow H^0 + h\nu \tag{1}$$

The probability of this happening to an H^+ in one second for an electron with velocity v is

$$P_{21}(v) = n_e \sigma_r(v) v \tag{2a}$$

When you integrate over a thermal distribution, you get

$$P_{21} = n_e \alpha \tag{2b}$$

where α is the *recombination coefficient*, a function of temperature.

Consider the reverse of reaction 1: it expresses the *photoionization* of the atom. In LTE, the rates of two processes must be equal by the principle of detailed balance. This means that one can derive the recombination rate α if you know the photoionization cross section. The details of this derivation are in Rybicki and Lightman Chap 10 or Osterbrock Appendix I. The derivation involves the following important points:

(1) First we set up the equation of *statistical equilibrium* between the two states (neutral and ionized). According to detailed balance, the total downward (recombination) rate equals the total upward (ionization) rate, where these total rates include spontaneous transitions (downward only) and induced transitions (both upward and downward).

The spontaneous transition downward for recombination is just the collisional interaction between the two particles—the same thing as the recombination rate we want to calculate. Induced upward transitions are the photoionization process. Induced downwards transitions are the inverse of the photoionization process and, in TE, their rate is $e^{-h\nu/kT}$ times the induced upwards rate (we will encounter this again later, and explain a bit more why). Both the photon and electron have continuous energy distributions, so we have to be a bit careful and write the detailed balance equation for each velocity interval and its corresponding frequency interval. So we write

$$\mathcal{R}(v \rightarrow v + dv) = (1 - e^{-h\nu/kT}) \mathcal{I}(\nu \rightarrow \nu + d\nu) \tag{3a}$$

where $(\mathcal{R}, \mathcal{I})$ are the recombination and photoionization rates, respectively. In terms of collisional cross sections, photoionization cross sections, and photon number density expressed in terms of $B_\nu(T)$ we have

$$n_e n_{H^0} \sigma_r(v) v f(v) dv = (1 - e^{-h\nu/kT}) n_H^+ \left(\frac{4\pi B_\nu(T)}{h\nu} \right) \sigma_{PI}(\nu) d\nu \tag{3b}$$

where σ_{PI} is the **P**hoto**I**onization cross section.

In recombination, energy conservation means that the the KE of the incoming electron is equal to the sum of the ionization energy IP and the energy of the emitted photon:

$$\frac{m_e v^2}{2} = h\nu_{IP} + h\nu \tag{4}$$

which leads to the differential relation between electron velocity interval and emitted frequency interval

$$m_e v \; dv = h \; d\nu \tag{5}$$

(2) Next we write the equation of *thermodynamical equilibrium* between the two states (neutral and ionized). This is just the Saha equation

$$\frac{n_e n_{H^+}}{n_{H^0}} = \frac{2g_+}{g_0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-h\nu_{IP}/kT} \tag{6}$$

where (g_+, g_0) are the statistical weights of the ionized and neutral state; recall that $g_0 = 2n^2$ (sum of l levels in a given n, times 2 for spin) and $g_+ = 1$ (it's a bare proton).

(3) The two above equations involve the same quantities, so we can derive the desired relationship between the recombination and photoionization cross sections

$$\sigma_r(v) = 2n^2\left(\frac{h\nu}{m_e c v}\right)^2 \sigma_{IP}(\nu) \tag{7}$$

(4) For H^0 and arbitrary n,

$$\sigma_{IP}(\nu) = \frac{\mathcal{G}\sigma_0}{n^5} \tag{8}$$

where $\mathcal{G} \approx 1$ is the Gaunt factor. Re-expressing equation 7 in terms of interpretable ratios, we can write

$$\sigma_r(v) = \left(\frac{h\nu_{IP}}{me^2}\right)\frac{h\nu_{IP}}{m_e v^2/2}\frac{\nu_{IP}}{\nu}\frac{\sigma_{IP}}{n^3} \tag{9}$$

where σ_{IP} is the photoionization cross section at the ionization threshold. The term in parenthesis equals 2.67×10^{-5}.

Consider a typical HII region with $T \sim 10^4$ K. The typical electron KE is $kT \sim 1$ ev and $\nu_{IP} = 13.6$ ev, so the second ratio ~ 13. The third ratio is necessarily < 1, but typically not by much. Finally, $\sigma_r = 7.91 \times 10^{-18}$ cm^2. Putting all this together, we get (for $n = 1$)

$$\sigma_r(v) \sim 3 \times 10^{-21} {T_4}^{-1} \text{ cm}^2 \tag{10}$$

where $T_4 = \frac{T}{10^4}$ K.

2. THE RECOMBINATION COEFFICIENT

From equation 2, we see that the recombination coefficient α is an integral over $\sigma_r(v)v$, so we expect $\alpha \propto T^{-0.5}$. Following Spitzer (p 107, Table 5.2), we write

$$\alpha^{(n)} = \frac{2.06 \times 10^{-11} Z^2}{T^{1/2}} \phi_n(\beta) \text{ cm}^3 \text{ s}^{-1} \tag{11}$$

where $\phi_n(\beta)$ includes the Gaunt factor and n can be 1 or 2 in Spitzer's parlance; it's given in Table 5.2. $\beta = \frac{h\nu_{LP}}{kT}$.

2.1. Approximation for $n = 1$

Spitzer's $n = 1$ case is the total recombination coefficieint summed over all levels. A good approximation is

$$\alpha_H^{(1)} = 4.02 \times 10^{-13} T_4^{-0.726} \text{ cm}^3 \text{ s}^{-1} \tag{12}$$

2.2. Approximation for $n = 2$

Spitzer's $n = 2$ case is the total recombination coefficieint summed over all levels *but excluding level 1.* A good approximation is

$$\alpha_H^{(2)} = 2.59 \times 10^{-13} \ T_4^{-0.845} \text{ cm}^3 \text{ s}^{-1} \tag{13}$$

Why is this case interesting? Because recombinations to level 1 don't last in a case B ("on-the-spot") case—the atoms are immediately re-ionized.

3. A SURPRISING POINT

From the above, you'd think that radiative recombinations to the high-n states that are involved with radio recombination lines are negligible. And you'd be right! So how can we possibly have observable radio recombination lines?

Surprise! This is one regime—perhaps the only one for the ISM—where *3-body interactions* are important. At high n, the atoms get huge and have huge collisional cross sections.