Astro 250 – Planetary Dynamics – Notes on Gauss's Perturbation Equations

1 What is an Osculating Element?

The osculating elements are those Keplerian elements of an ellipse (a, e, i, $\tilde{\omega}$, Ω, f) that a particle *would* have at a particular instant in time, *if* the potential were that of a point mass. It refers to the *instantaneous ellipse* which is *fitted* to the particle's *instantaneous position and velocity* with respect to the central mass. In other words, given a position and a velocity at an instant in time, I can always fit an instantaneous Kepler ellipse; the osculating elements are the elements of that ellipse.

For a point-mass (Kepler or $1/r$) potential, the osculating elements are just the Keplerian elements of the orbit traced out by the particle. In this (boring) case, all the osculating elements (except for f) do not change with time.

By contrast, this class is concerned with *slightly non-point-mass* potentials: potentials where the central point mass dominates, but where there is a small perturbation from something else—maybe another planet, or a disk, or non-sphericity of the central mass (rotational or tidal bulges), or anisotropic radiation fields, or even an external wind from the interstellar medium. *For such slightly non-Keplerian potentials ($1/r$ plus a little bit more), the trajectory of the particle is still most easily described as that of an ellipse, but an ellipse whose elements are slowly changing with time.* In other words, the Keplerian ellipse slowly "morphs" with time. The osculating elements are the particle's *instantaneous* Keplerian elements at a given time t. They are just another way of expressing $(x, y, z, \dot{x}, \dot{y}, \dot{z})$, but they enable us to visualize more readily the particle's trajectory. Ask yourself which you would prefer to be told: $[x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)]$ or $[a(t), e(t), i(t), \tilde{\omega}(t), \Omega(t), f(t)]$ (we can even drop $f(t)$ from consideration if we are not interested in the particle's exact position, which we often are not because we just want to know how the general *shape and orientation* of the orbit is morphing with time).

2 Gauss's Perturbation Equations for Osculating Elements

For $d\vec{F} = R\hat{r} + T\hat{\theta} + N\hat{z}$ applied at r and f:

$$\dot{a} = \frac{2a^{3/2}}{\sqrt{G(m_1 + m_2)(1 - e^2)}} \left[Re \sin f + T(1 + e \cos f) \right]$$

[Editorial remark: I prefer the following version of $\dot{e}$ which uses f instead of E (eccentric anomaly, which I leave for MD to describe)]:

$$\dot{e} = \sqrt{\frac{a(1-e^2)}{G(m_1+m_2)}}\left[R\sin f + T\left(\frac{2\cos f + e(1+\cos^2 f)}{1+e\cos f}\right)\right]$$

$$\dot{i} = \sqrt{\frac{a(1-e^2)}{G(m_1+m_2)}}\frac{N\cos(\omega+f)}{(1+e\cos f)}$$

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{G(m_1+m_2)}}\frac{N\sin(\omega+f)}{\sin i(1+e\cos f)}$$

Note the following is $\dot{\tilde{\omega}}$ not $\dot{\omega}$ as given in MD:

$$\dot{\tilde{\omega}} = \frac{1}{e}\sqrt{\frac{a(1-e^2)}{G(m_1+m_2)}}\left[-R\cos f + T\sin f\left(\frac{2+e\cos f}{1+e\cos f}\right)\right] + \frac{1-\cos i}{\sin i}\sqrt{\frac{a(1-e^2)}{G(m_1+m_2)}}\frac{N\sin(\omega+f)}{1+e\cos f}$$