Astro 250: Solutions to Problem Set 1

by Eugene Chiang

Problem 1. *Apsidal Line Precession*

A satellite moves on an elliptical orbit in its planet's equatorial plane. The planet's gravitational potential has the form

$$U \approx \frac{GM_p}{r}\left[1 - J_2\left(\frac{R_p}{r}\right)^2 P_2(\cos\theta)\right], \tag{1}$$

where r is the distance from the planet to the satellite, θ is the polar angle measured from the planet's spin axis, $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ is the Legendre polynomial of degree 2, M_p and R_p are the planet's mass and radius, respectively, G is the gravitational constant, and J_2 is a constant that characterizes the dimensionless strength of the quadrupole field of the planet (the degree of planetary oblateness). Celestial mechanicians define their potentials U to be positive, by contrast with the usual convention in physics.

a) *Use the appropriate perturbation equation due to Gauss (equation 2.165 of MD) to calculate $\langle\dot{\tilde{\omega}}\rangle$, the time-averaged precession rate of the satellite's apsidal line.*

b) *Show that a and e do not suffer any time-averaged variations using the appropriate equations of Gauss (also found in section 2.9 of MD).*

a) First, some relations for Keplerian ellipses to lowest order in e that are convenient to have at one's fingertips:

$$r = a(1 - e\cos f) \tag{2}$$

$$\dot{f} = n(1 + 2e\cos f) \tag{3}$$

$$GM_p = n^2 a^3 \tag{4}$$

$$\dot{r} = nae\sin f \tag{5}$$

The perturbation equations for $\tilde{\omega}$, e, and a in the plane read, again to lowest order in e:

$$\frac{d\tilde{\omega}}{dt} = \frac{1}{nae}(-R\cos f + 2S\sin f) \tag{6}$$

$$\frac{de}{dt} = \frac{1}{na}(R\sin f + 2S\cos f) \tag{7}$$

$$\frac{da}{dt} = \frac{2}{n}\left[Re\sin f + S(1 + e\cos f)\right] \tag{8}$$

One fact worth remembering is that when you pull inwards ($R < 0$) on a particle near its periapse ($f \approx 0$), its apsidal line advances ($\frac{d\tilde{\omega}}{dt} > 0$; $\tilde{\omega}$ advances in the direction of increasing true anomaly).

The planet's axisymmetric bulge causes a purely inward radial perturbation force in its equatorial plane. Using $P_2(0) = -1/2$, and remembering that celestial mechanicians define the potential to be positive, we have:

$$R = \frac{d}{dr}\frac{JGM_p}{2r}\left(\frac{R_p}{r}\right)^2 = -\frac{3JGM_p}{2r^2}\left(\frac{R_p}{r}\right)^2 \tag{9}$$

We insert this into perturbation equation (6) and average over 1 orbit to find the time-averaged precession rate.

$$< \frac{d\tilde{\omega}}{dt} >= \frac{3JR_p^2GM_p}{2nae}\frac{\int_0^{2\pi}\frac{\cos f}{r^4}\frac{dt}{df}df}{\int_0^{2\pi/n}dt} \tag{10}$$

Note that we must average over time, not true anomaly; the particle spends more time near apoapse than periapse, and the perturbation force must be weighted to account for this. Insert equations (2–4) into (9) to find

$$\boxed{< \frac{d\tilde{\omega}}{dt} >= \frac{3\sqrt{GM_p}JR_p^2}{2}a^{-7/2}} \tag{11}$$

Physically, the $1/r^4$ perturbation force is much stronger at periapse than apoapse, so that despite the extra time spent at apoapse, the attractive perturbation force is felt principally at periapse. The result of the net extra inward tug at periapse is that the apsidal line advances.

b) The radial force R in the perturbation equations for a and e are weighted by $\boxed{\sin f}$, $\boxed{\text{which averages to zero over 1 orbit}}$.

Problem 2. *The Perversity of Osculating Elements*

Deduce the values and time dependences of the osculating elements that characterize a perfectly circular equatorial orbit of radius r around an oblate planet. Employ the potential given by equation (1) above. To get started, compute the relation between the *angular velocity and the orbital radius. Remember that the osculating elements are those elements of a Keplerian ellipse that just "kisses" (is instantaneously tangential to) the actual position and velocity of the particle. We fit the Kepler orbit to the motion by assuming that the particle moves in a point-mass potential. Here the potential is not that of a point-mass, but we wish to describe the motion of the particle as if it were.*

The osculating elements of a particle at a particular instant in time are the a, e, and $\tilde{\omega}$ appropriate to a Keplerian (read: point-mass potential) ellipse fitted to the particle's motion at that instant. Here the particle is executing a perfect circle about a non-point-mass potential, and we (perversely) wish to describe its circular orbit in terms of constantly changing Keplerian ellipses. It is important to remember that the n, a and e appropriate to the fitted ellipse at any instant are *abstractions* which fall out of the osculating element formalism and do not correspond to anything terribly physical.

Let's first solve for a and e as a function of r (and quadrupole coefficient J). Two unknowns call for two equations. The first equation is a geometrical constraint: since the particle is moving purely azimuthally in its perfect circle, the particle must be either at the periapse or the apoapse of the fitted ellipse. Let's guess that the particle is at the periapse (if we choose incorrectly, the fitted e will turn out to be negative):

$$r = a(1 - e) \tag{12}$$

The second equation fits the circular velocity; equate the centripetal acceleration to the total radial acceleration due to the planet:

$$\begin{aligned} r\dot{\lambda}^2 &= \frac{GM_p}{r^2}\left[1 + \frac{3}{2}J\left(\frac{R_p}{r}\right)^2\right] \\ &= rn^2(1 + 4e\cos 0) \end{aligned} \tag{13}$$

where for the last expression we have used (3). Use (12–13) to solve for a and e in terms of r, remembering that for osculating elements, $n^2a^3 = GM_p$ always (the fitted ellipse is a Keplerian ellipse):

$$\boxed{e = \tfrac{3}{2}J\left(\tfrac{R_p}{r}\right)^2} \tag{14}$$

$$\boxed{a = r(1 + \tfrac{3}{2}J\left(\tfrac{R_p}{r}\right)^2)} \tag{15}$$

Now for $\tilde{\omega}$ we argue as follows: the only way an ellipse can trace out a circle is if its longitude of periapse advances as quickly as its mean longitude, $\dot{\tilde{\omega}} = \dot{\lambda}$. Then

$$\boxed{\tilde{\omega} = \dot{\lambda}t + \tilde{\omega}_0} \tag{16}$$

where $\dot{\lambda}$ can be expressed in terms of r using equation (13).

Since $\dot{a}$ and $\dot{e}$ are both proportional to $R\sin f$, and since the particle is always at $f = 0$, $\dot{a} = \dot{e} = 0$. Accordingly, the values for a and e that we deduced have no time dependences.

We can use Gauss's equations to prove that $\dot{\tilde{\omega}} = \dot{\lambda}$. Using (6),

$$\begin{aligned} \frac{d\tilde{\omega}}{dt} &= \frac{1}{nae}\frac{3GM_pJR_p^2}{2r^4} \\ &= \frac{GM_p}{nar^2} \\ &= na^2/r^2 \\ &= n(1+2e) = \dot{\lambda} \end{aligned} \tag{17}$$

Problem 3. *Velocity Ellipsoid in Collisionless Keplerian Disks*

Consider a circumstellar disk composed of massless test particles which move without colliding on orbits of eccentricity $e \ll 1$*. What is the ratio of the velocity dispersions in the radial and azimuthal directions? Material in the reading from Binney & Tremaine (1987) is relevant to this problem.*

By velocity dispersion in an axisymmetric disk we mean the following. Imagine ourselves co-rotating with the disk on a circular orbit. At a given instant in time, we measure the apparent velocities of all particles whizzing by our position. We then (1) square and (2) average the apparent velocities in a particular direction to obtain the squared velocity dispersion, σ^2*, in that direction. The problem asks you to obtain the ratio of squared dispersions in the radial and azimuthal directions,* σ_r^2/σ_ϕ^2*. Provided the disk is collisionless (particle pass through each other), this ratio is magically independent of the actual distribution of random velocities; i.e., this ratio is independent of the actual distribution of eccentricities, provided they are small.*

To begin, let us fix our orbital radius at R_0 and our angular velocity at Ω_0. The particles which manage to cross our position (so that we can measure their velocities) originate from *a range of parent guiding centers* centered about our own orbit. Those particles which come from relatively distant guiding centers will have relatively high eccentricities in order to reach us. Consider a particle which crosses our position from a (generic) guiding center at R_g which is a radial distance $x_g > 0$ inside our fixed, circular orbit: $R_g + x_g = R_0$. The orbit of the particle is expressed in terms of its radial and azimuthal excursions, x and y, from its circular guiding center orbit, viz.

$$x = X\cos(\kappa t + \psi) = X\cos(nt + \psi) \tag{18}$$

$$y = Y\sin(nt + \psi) = -2X\sin(nt + \psi) \tag{19}$$

where the rightmost equalities are appropriate for Keplerian ellipses of small eccentricity. In this regime, the radial epicyclic frequency, κ, equals the azimuthal frequency, n, so that orbits are closed. Moreover, the amplitude of excursions in the azimuthal direction, Y, is twice that in the radial direction, X.

Calculate first the radial velocity dispersion, σ_R^2. In the following, the overhead $\overline{\text{bar}}$ denotes an average over all particles crossing our position at a given instant in time, t_0. It is an average over all particles originating from all contributing guiding centers.

$$\begin{aligned}
\sigma_R^2 &= \overline{\dot{x}^2} \\
&= \overline{X^2 n^2 \sin^2(nt_0 + \psi)} \\
&= n^2 \overline{X^2 \sin^2(nt_0 + \psi)} \\
&= \frac{n^2}{2}\overline{X^2}
\end{aligned} \tag{20}$$

The last equality follows from our assumption that amplitudes and phases of measured particles are completely uncorrelated. (For two uncorrelated variables, $\overline{AB} = \overline{A} \times \overline{B}$.) We can go no further without knowing the explicit distribution of eccentricities and semi-major axes of the particles. However, this information is not required because the question asks only for the ratio of σ_R^2 to σ_θ^2, and the quantity $\overline{X^2}$ will divide out in that ratio, as we show below.

Calculate now the azimuthal velocity dispersion, remembering that the apparent azimuthal velocity of a particle (the one that you measure) is really the difference between its inertial space azimuthal velocity and your own inertial space circular velocity. Realize below that in certain instances the difference between R_0 and R_g is negligibly small.

$$\begin{aligned}
\sigma_\theta^2 &= \overline{(\dot{\theta}R_0 - \Omega_0 R_0)^2} \\
&= R_0^2\overline{(\dot{\theta} - \Omega_0)^2} \\
&= R_0^2\overline{\left[(\dot{\theta} - \Omega_g) + (\Omega_g - \Omega_0)\right]^2}
\end{aligned}$$

$$= R_0^2 \overline{\left(\frac{\dot{y}}{R_g} - \frac{d\Omega}{dR} x_g\right)^2}$$

Now realize that x_g, the radial distance from the guiding center to your position, is merely x, the instantaneous radial excursion of the particle away from its guiding center (for some reason, this seemingly obvious fact took me 3 hours yesterday to realize):

$$\begin{aligned}
&= R_0^2 \overline{\left[\frac{-2Xn\cos(nt_0+\psi)}{R_g} - \frac{d\Omega}{dR} X\cos(nt_0+\psi)\right]^2} \\
&\approx \overline{\left[-2Xn\cos(nt_0+\psi) + \frac{3n}{2} X\cos(nt_0+\psi)\right]^2} \\
&\approx \frac{n^2}{2} \overline{X^2 (-2+\frac{3}{2})^2} \\
&= \frac{n^2}{8} \overline{X^2}
\end{aligned} \tag{21}$$

Then dividing (20) by (21) and taking the square root, we obtain our final answer:

$$\boxed{\frac{\sigma_R}{\sigma_\theta} = 2} \tag{22}$$

which is the inverse of the ratio of Keplerian epicyclic motions about a given guiding center. It is different from that ratio because of the underlying mean *shear* of the disk. Of course, the squared ratio is 4. Note that the velocity dispersion in the z direction is de-coupled from the planar components (inclinations have nothing to do with eccentricities, provided both are small). The squared ratio of 4-to-1 (radial component being larger) for Keplerian disks figures prominently in studies of disks; Chandrasekhar was the first to derive this ratio, I believe; different values will obtain for different rotation (shearing) profiles in disks, be they Galactic or planetary; in disks composed of perfectly spherical, inelastic, collisional particles, the velocity ellipsoid tends to round itself into a sphere (i.e., for optical depth $\tau \gg 1$, collision rates become so high that the particles act like a gas of isotropic pressure, $\sigma_r^2 = \sigma_\phi^2 = \sigma_z^2$). The cross component $\sigma_{r\phi}^2 \equiv \langle v_r v_\phi \rangle$ determines the rate of angular momentum transport across annuli—i.e., radial mass transport—in accretion disks (it is zero in our collisionless problem).