## Astro 250 – Planetary Dynamics – Problem Set 2

## Do at least 1 out of 3.

Readings: Murray & Dermott 3.2–3.6, 3.8, 3.9 (up through page 98), 3.11. Optional: section 3.3.2 of BT up to equation (3.124), keeping in mind that BT's treatment in Chapter 3 is meant for generic bar potentials and is not specific to the circular restricted 3-body problem. Also optional: BT section 8.2 up through "case (c) adiabatic invariance," and section 8.3 up through equation (8.92).

### Problem 1. Impulse approximation

Consider close encounters between a test particle and the secondary mass $m_2$ in the restricted 3-body problem with small mass ratio between the secondary and the primary, $\mu \ll 1$. Take the secondary mass to occupy a perfectly circular orbit of radius $a_0 = 1$. For parts (a)–(d), assume that the test particle is inserted at opposition on a very nearly circular orbit with semi-major axis $a = 1 - x$ and that $\mu^{1/3} \ll x \ll 1$.

a) How long does it take the test particle to move an azimuthal distance $2x$ relative to $m_2$? Estimate the time rate of change of $x$, $\dot{x}$, during the encounter by calculating the radial impulse the test particle receives when it moves past $m_2$ on an unperturbed orbit.

b) Estimate the eccentricity, $\Delta e$, that results from the initial encounter of the test particle with $m_2$. Neglect the Coriolis acceleration which only introduces a numerical factor of order unity. Express $\Delta e$ as a function of $\mu$ and $x$.

c) Use the Jacobi constant to estimate the change in semi-major axis, $\Delta a$, that results from the encounter. You may find it helpful to read the section in the text on the Tisserand relation before attempting this part. Express $\Delta a$ as a function of $\mu$ and $x$ and include its sign.

d) What is the change in inertial space angular momentum, $\Delta h$, suffered by the test particle in this initial encounter? Express $\Delta h$ as a function of $\mu$ and $x$ and include its sign.

e) Calculate the radial spacing, $\delta a$, between the location of neighboring principal mean motion resonances. Each resonance is characterized by a positive integer $p$. Consecutive encounters of an unperturbed test particle moving on the $p^{th}$ resonant orbit occur every $p$ orbital periods of the secondary mass. Express $\delta a$ as a function of $p$.

f) Find the critical $x$ at which $\Delta a = \delta a$. Express $x_{\text{crit}}$ as a function of $\mu$. This expression can be compared with Wisdom's resonance overlap criterion for chaos, a topic we will cover later.

g) Assume that at each subsequent encounter with $m_2$, the test particle's angular momentum changes by the amount $\Delta h$ calculated in part (d). Calculate an approximate expression for the time-averaged torque, $T$, on the test particle. Express $T$ as a function of $\mu$ and $x$ and include its sign. This expression is useful in studies of ring shepherding; can you see why?

**Problem 2.** Tadpoles and Horseshoes

Consider the circular restricted three-body problem. Start from the equation of motion of the test particle, expressed in polar co-ordinates in the co-rotating frame:

$$\ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta} = \frac{\partial U}{\partial r} \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = \frac{1}{r}\frac{\partial U}{\partial \theta}$$

where $U$ is the celestial mechanician's potential in the rotating frame (the so-called "pseudo-potential"):

$$U = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}r^2 .$$

Take $\mu \ll 1$. Here $r$ is the distance of the particle to the center of mass, $r_1$ is the distance of the particle to the primary of mass $1-\mu$, and $r_2$ is the distance of the particle to the secondary of mass $\mu$. The secondary executes a perfectly circular orbit of radius 1 from the primary at an angular frequency of 1.

Describe the position of the test particle in terms of its radial deviation away from the unit circle: $\Delta = r - 1 \ll 1$. We will derive an equation for the shapes of those orbits that librate about the $L_4$ and $L_5$ points—so-called tadpole and horseshoe orbits. We will also derive an expression for the libration periods of small tadpole orbits. To filter out the fast epicyclic motion and select only the *slow* motion of libration about $L_4$ and $L_5$, take $d/dt \ll 1$.

a) Expand the potential retaining terms of order $\Delta$, $\Delta^2$, and $\mu$. (Start from the law of cosines to write down expressions for $r_1$ and $r_2$.)

b) Show that to leading order in the radial component of the equation of motion,

$$3\Delta + 2\dot{\theta} \approx 0.$$

c) Show that to leading order in the azimuthal component of the equation of motion,

$$\ddot{\theta} \approx -\frac{3}{2}\mu\frac{\partial}{\partial\theta}\left[\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2)\right],$$

where $\theta < 2\pi$ so that $\sin(\theta/2) > 0$.

d) Derive the integral relation

$$\Delta^2 + \frac{4}{3}\mu\left[\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2)\right] = 4\mu B,$$

where $B$ is a constant of integration. This equation yields the shape of the tadpole/horseshoe orbit. Whether the orbit is a tadpole or horseshoe depends on the value of $B$.

e) What value of $B = B_0$ corresponds to the triangular equilibrium points, $\theta = \pi/3$ ($L_4$) and $\theta = 5\pi/3$ ($L_5$)?

f) What value of $B = B_1$ corresponds to the maximal tadpole orbit, i.e., the tadpole orbit which extends to $L_3$? How close does this orbit get to the secondary? What is its maximum radial width? For $B > B_1$, the orbit is a horseshoe that encircles $L_3$.

g) What value of $B = B_2$ corresponds to the maximal horseshoe orbit, i.e., the horseshoe orbit that approaches the Hill sphere of the secondary? For these orbits, $\theta$ and $2\pi - \theta$ achieve minimum values equal to $F\mu^{1/3}$ where $F$ is a constant of order unity. What is the maximum radial width of these orbits?

h) For $B_0 < B < B_1$, calculate the endpoints of *small* tadpole orbits, i.e., those values of $\theta$ where $\dot{\theta} = 0$, for tadpole orbits which never stray far from the Lagrange point about which they librate. Use the relations under (b) and (d). Express the endpoint locations in terms of $B$.

i) Use (b), (d), and (h) to derive an expression for the (slow) period of libration of *small* tadpole orbits. You will need to expand the expression in brackets in (d) about $\theta = \pi/3$ or $\theta = 5\pi/3$. Your expression should not depend on the size of the tadpole in the small tadpole limit. Evaluate this libration period for a Trojan asteroid co-orbiting with Jupiter.

**Problem 3.** Isolation of Planetary Embryos

Consider a disk composed of massive planetesimals. The most massive planetesimal has the largest cross-section for accreting other bodies, not only because it has the largest geometric radius but also because it possesses the largest gravitational focussing factor. Thus, the most massive body in the swarm tends to accrete all other bodies in its vicinity. This problem computes the point at which this initial feeding frenzy stops.

A body of mass $M$ at distance $r$ from a star of mass $M_*$ can accrete other bodies within a few Hill radii of its orbit:

$$\Delta r = \pm B(M/3M_*)^{1/3} r$$

where numerical experiments demonstrate that $B \approx 2.5$ for a dynamically cold disk of planetesimals (Greenberg et al. 1991, 94, 98). Take $\sigma$ to be the surface mass density (mass per unit face-on area) of the planetesimal disk. Derive an expression for the "isolation mass," the maximum mass which can accrete within such a disk at every radius. Evaluate the isolation mass, in units of an Earth mass, as a function of disk radius for conditions appropriate to the minimum-mass solar nebula: $\sigma \sim 20(r/\mathrm{AU})^{-3/2}\ \mathrm{g/cm^2}$, $M_* = M_\odot$.