Astro 250: Solutions to Problem Set 2

by Eugene Chiang

Problem 1. *Impulse approximation*

Consider close encounters between a test particle and the secondary mass m_2 in the restricted 3-body problem with small mass ratio between the secondary and the primary, $\mu \ll 1$. Take the secondary mass to occupy a perfectly circular orbit of radius $a_0 = 1$. For parts (a)–(d), assume that the test particle is inserted at opposition on a very nearly circular orbit with semi-major axis $a = 1 - x$ and that $\mu^{1/3} \ll x \ll 1$.

a) How long does it take the test particle to move an azimuthal distance $2x$ relative to m_2? Estimate the time rate of change of x, $\dot{x}$, during the encounter by calculating the radial impulse the test particle receives when it moves past m_2 on an unperturbed orbit.

For this problem, we'll take a_0 and n_0 to be the fixed semi-major axis and mean motion, respectively, of the secondary mass, m_2. In natural units, a_0, n_0 and the gravitational constant G are set to one (which implies that $m_1 = 1$ and $\mu = m_2$). Usually we'll write out all these variables explicitly to make physical sense of our equations and to check units, but occasionally we'll get rid of them to make the algebra appear cleaner.

The difference in angular velocities between the test particle and m_2 is

$$\Delta n = -\frac{d\Omega}{dr} x = \frac{3n_0}{2a_0} x \tag{1}$$

Solve for the time, t_{2x}, required to move $2x$:

$$a_0 \Delta n \times t_{2x} = 2x \tag{2}$$

$$\Longrightarrow \boxed{t_{2x} = \tfrac{4}{3n_0} = \tfrac{4}{3}} \tag{3}$$

independent of x and equal to $2P_0/3\pi$, where P_0 is the orbital period of m_2.

In the crude impulse approximation, say that the test particle receives a radial velocity kick outwards, $\dot{x} > 0$, equal to the gravitational acceleration due to m_2 at closest approach, times the interaction time t_{2x}:

$$\boxed{\dot{x}} \sim \frac{G\mu}{x^2} \times t_{2x} \sim \boxed{\tfrac{4G\mu}{3n_0 x^2} = \tfrac{4\mu}{3x^2}} \tag{4}$$

b) Estimate the eccentricity, Δe, that results from the initial encounter of the test particle with m_2. Neglect the Coriolis acceleration which only introduces a numerical factor of order unity. Express Δe as a function of μ and x.

The radial velocity kick of part (a) manifests itself as a new eccentricity. To order of magnitude,

$$\dot{x} \sim n_0 a_0 \Delta e \tag{5}$$

$$\Longrightarrow \boxed{\Delta e \sim \tfrac{4}{3}\mu\left(\tfrac{a_0}{x}\right)^2} \tag{6}$$

c) Use the Jacobi constant to estimate the change in semi-major axis, Δa, that results from the encounter. You may find it helpful to read the section in the text on the Tisserand relation before attempting this part. Express Δa as a function of μ and x and include its sign.

Neglecting unimportant overall multiplicative factors, we write the Jacobi constant as

$$C_J = E_T - n_0 h \tag{7}$$

where E_T is the energy per unit mass of the test particle measured in inertial space, and h is the inertial space specific angular momentum. Take E_T and h to be mostly due to motion about the largest mass, m_1; this is the route towards deriving Tisserand's relation. We know for orbits about m_1 that

$$E_T = -\frac{Gm_1}{2a} = -\frac{Gm_1}{2a_0(1-\frac{x}{a_0})} \tag{8}$$

and that

$$\begin{aligned} h &= h|_{periapse} = na\sqrt{\frac{1+e}{1-e}} \times a(1-e) \\ &= na^2\sqrt{1-e^2} \\ &= n_0 a_0^2 \left(1-\frac{x}{a_0}\right)^{-3/2}\left(1-\frac{x}{a_0}\right)^2\sqrt{1-e^2}\,. \end{aligned} \tag{9}$$

Substitute (8) and (9) into C_J, use $n_0^2 a_0^3 = Gm_1$, divide out the overall multiplicative factor of Gm_1, and omit all constants such as a_0 and n_0 to obtain:

$$C_J = \frac{1}{2(1-x)} + (1-x)^{1/2}\sqrt{1-e^2} + O(\mu) \tag{10}$$

Drop the order μ term and expand to *second order* in x, in anticipation of the fact that first order terms will vanish:

$$\begin{aligned} C_J &= \frac{1}{2}(1+x+x^2) + (1 - \frac{1}{2}x - \frac{1}{8}x^2)\sqrt{1-e^2} \\ &= \frac{3}{2} + \frac{3}{8}x^2 - \frac{1}{2}e^2 \end{aligned} \tag{11}$$

Now use the constancy of the Jacobi integral before (no prime) and after (prime) the encounter:

$$\begin{aligned} C_J' - C_J = 0 = \frac{3}{8}\left[(x+\Delta x)^2 - x^2\right] - \frac{1}{2}(\Delta e)^2 \\ \Longrightarrow \Delta x = \frac{2}{3}\frac{(\Delta e)^2}{x} \end{aligned} \tag{12}$$

Since $a = 1-x$, $\Delta a = -\Delta x$. Plugging in (6), we finally have

$$\boxed{\frac{\Delta a}{a_0} = -\frac{32}{27}\mu^2\left(\frac{a_0}{x}\right)^5} \tag{13}$$

Note the sign—the close encounter with the secondary mass has reduced the semi-major axis of the particle. If the test particle were placed initially outside the orbit of the secondary mass, then $x < 0$ and $\Delta a > 0$. The secondary mass acts to repel the test particle and to excite its eccentricity.

d) What is the change in inertial space angular momentum, Δh, suffered by the test particle in this initial encounter? Express Δh as a function of μ and x and include its sign.

It is sufficient for this question to work to first order in x. Using relations for h above,

$$\begin{aligned} \boxed{\Delta h} = h' - h &= n_0 a_0^2\left[1 - \frac{1}{2}\frac{x+\Delta x}{a_0}\right]\sqrt{1-\frac{1}{2}(\Delta e)^2} - n_0 a_0^2(1 - \frac{1}{2}\frac{x}{a_0}) \\ &= -\frac{1}{2}(\Delta e)^2 n_0 a_0^2 - \frac{1}{2}\frac{\Delta x}{a_0} n_0 a_0^2 \end{aligned}$$

$$= \quad -\frac{1}{2} n_0 a_0^2 \left[(\Delta e)^2 + \frac{2}{3} (\Delta e)^2 \frac{a_0}{x} \right]$$

$$\approx \quad \boxed{-\tfrac{16}{27} n_0 a_0^2 \, \mu^2 \left(\tfrac{a_0}{x}\right)^5} \tag{14}$$

The torque due to the outside secondary mass is negative, causing the test particle to lose angular momentum. The reverse would be true if the test particle were outside the secondary mass.

e) Calculate the radial spacing, δa, between the location of neighboring principal mean motion resonances. Each resonance is characterized by a positive integer p. Consecutive encounters of an unperturbed test particle moving on the p'th resonance orbit occur at intervals of p periods of the relative orbit of the massive bodies. Express δa as a function of p.

First find the x_p appropriate to the p^{th} resonance. By definition,

$$\frac{2\pi}{n_0} p = \frac{2\pi}{n_p - n_0} \tag{15}$$

where n_p is the mean motion at the p^{th} resonance. Equation (15) gives what is called the *synodic period*, P_{synodic}, the period between successive encounters. Let $n_p = n_0(1 + x_p/a_0)^{-3/2}$. Insert this relation into the above equation to find

$$\frac{x_p}{a_0} = (1 - \frac{1}{p})^{-2/3} - 1 \approx \frac{2}{3p} \tag{16}$$

where the latter approximation holds in the $p \gg 1$ limit; higher order resonances are found closer to the secondary mass (corotation circle). Now $\delta a_{p,p+1} \equiv x_p - x_{p+1}$ so we find that

$$\boxed{\tfrac{\delta a_{p,p+1}}{a_0} = (1 - \tfrac{1}{p})^{-2/3} - (1 - \tfrac{1}{p+1})^{-2/3} \approx \tfrac{2}{3p(p+1)}} \tag{17}$$

where again, the latter approximation holds in the $p \gg 1$ limit.

f) Find the critical x at which $\Delta a = \delta a$. Express x_{crit} as a function of μ. This expression can be compared with Wisdom's resonance overlap criterion for chaos, a topic we will cover later.

Work in the $p \gg 1$ limit. Then

$$|\Delta a| = |\delta a|$$

$$\Longrightarrow \frac{32}{27}\mu^2\left(\frac{a_0}{x_{\rm crit}}\right)^5 = \frac{2}{3p_{\rm crit}^2}$$

$$= \frac{3}{2}\left(\frac{x_{\rm crit}}{a_0}\right)^2$$

$$\Longrightarrow \boxed{\frac{x_{\rm crit}}{a_0} = \left(\frac{8\mu}{9}\right)^{2/7}} \tag{18}$$

where we have used (16), (17), and (13). For $x < x_{\rm crit}$, the perturbation can kick the test particle across resonances, leading to chaotic trajectories.

g) Assume that at each subsequent encounter with m_2, the test particle's angular momentum changes by the amount Δh calculated in part (d). Calculate an approximate expression for the time-averaged torque, T, on the test particle. Express T as a function of μ and x and include its sign. This expression is useful in studies of ring shepherding; can you see why?

Keep working in the $p \gg 1$ limit.

$$\begin{aligned}\boxed{T} &\sim \Delta h/P_{\rm synodic} \\ &\sim \frac{\Delta h\, n_0}{2\pi p} \\ &\sim \frac{3\Delta h\, n_0|x|}{4\pi a_0} \\ &\sim \boxed{-\frac{4\,sign(x)}{9\pi}\frac{Gm_1}{a_0}\mu^2\left(\frac{a_0}{x}\right)^4}\end{aligned} \tag{19}$$

Problem 2. *Tadpoles and Horseshoes*

Consider the circular restricted three-body problem. Start from the equation of motion of the test particle, expressed in polar co-ordinates in the co-rotating frame:

$$\ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta} = \frac{\partial U}{\partial r} \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = \frac{1}{r}\frac{\partial U}{\partial \theta}$$

where U is the celestial mechanician's potential in the rotating frame (the so-called "pseudo-potential"):

$$U = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}r^2 .$$

Take $\mu \ll 1$. Here r is the distance of the particle to the center of mass, r_1 is the distance of the particle to the primary of mass $1-\mu$, and r_2 is the distance of the particle to the secondary of mass μ. The secondary executes a perfectly circular orbit of radius 1 from the primary at an angular frequency of 1.

Describe the position of the test particle in terms of its radial deviation away from the unit circle: $\Delta = r - 1 \ll 1$. We will derive an equation for the shapes of those orbits that librate about the L_4 and L_5 points—so-called tadpole and horseshoe orbits. We will also derive an expression for the libration periods of small tadpole orbits. To filter out the fast epicyclic motion and select only the slow motion of libration about L_4 and L_5, take $d/dt \ll 1$.

a) Expand the potential retaining terms of order Δ, Δ^2, and μ. (Start from the law of cosines to write down expressions for r_1 and r_2.)

The celestial mechanician's potential in the rotating frame reads, in its full glory:

$$U = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}r^2 \tag{20}$$

Use the law of cosines to solve for r_1 and expand, dropping small terms:

$$\begin{aligned} r_1^2 &= r^2 + \mu^2 - 2r\mu\cos(\pi-\theta) && (21)\\ &\approx 1 + \Delta^2 + 2\Delta + 2\mu\cos\theta && (22)\\ r_1^{-1} &\approx 1 + \Delta^2 - \Delta - \mu\cos\theta\,. && (23) \end{aligned}$$

Similarly,

$$\begin{aligned} r_2^2 = r^2 + 1^2 - 2r\cos\theta && (24)\\ \Longrightarrow \frac{\mu}{r_2} \approx \frac{\mu}{\sqrt{2(1-\cos\theta)}}\,. && (25) \end{aligned}$$

And since $\frac{1}{2}r^2 = \frac{1}{2}(1 + 2\Delta + \Delta^2)$,

$$\boxed{U \approx \tfrac{3}{2} + \mu\left(\tfrac{1}{\sqrt{2(1-\cos\theta)}} - \cos\theta - 1\right) + \tfrac{3}{2}\Delta^2} \tag{26}$$

b) Show that to leading order in the radial component of the equation of motion,

$$3\Delta + 2\dot{\theta} \approx 0. \tag{27}$$

Since $\partial r = \partial\Delta$, $dU/dr = 3\Delta$. Then the radial component of the equation of motion reads:

$$\ddot{\Delta} - (1+\Delta)\dot{\theta}^2 - 2(1+\Delta)\dot{\theta} = 3\Delta\,. \tag{28}$$

Use the information provided in the problem to drop all but the leading order terms:

$$\boxed{-2\dot{\theta} = 3\Delta}. \tag{29}$$

c) Show that to leading order in the azimuthal component of the equation of motion,

$$\ddot{\theta} \approx -\frac{3}{2}\mu\frac{\partial}{\partial\theta}\left[\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2)\right], \tag{30}$$

where $\theta < 2\pi$ *so that* $\sin(\theta/2) > 0$.

Keep only leading order terms in the azimuthal equation to write:

$$\ddot{\theta} + 2\dot{\Delta} \approx \mu\frac{\partial}{\partial\theta}\left[-\cos\theta + [2(1-\cos\theta)]^{-1/2}\right] \tag{31}$$

Recall that $\sin^2\phi = (1-\cos 2\phi)/2$ to rewrite the terms in the big square brackets. Then take the time derivative of (29) to find $\dot{\Delta} = -2\ddot{\theta}/3$. Insert to find the equation desired.

d) Derive the integral relation

$$\Delta^2 + \frac{4}{3}\mu\left[\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2)\right] = 4\mu B, \tag{32}$$

where B is a constant of integration. This equation yields the shape of the tadpole/horseshoe orbit. Whether the orbit is a tadpole or horseshoe depends on the value of B.

Multiply (30) through by $\dot{\theta}$:

$$\dot{\theta}\ddot{\theta} \approx -\frac{3}{2}\mu\frac{\partial\theta}{\partial t}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2)\right) \tag{33}$$

Integrate over time:

$$\frac{1}{2}\dot{\theta}^2 + \frac{3}{2}\mu\left(\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2)\right) = \text{constant} \tag{34}$$

Use (29) to replace $\dot{\theta}$ with Δ, and let the final constant of integration equal $4\mu B$. The result is the equation desired.

e) What value of $B = B_0$ corresponds to the triangular equilibrium points, $\theta = \pi/3$ (L_4) and $\theta = 5\pi/3$ (L_5)?

At L_4 and L_5, $\Delta = 0$ and $\theta = \pi/3, 5\pi/3$. Plug these values into (32) to find $\boxed{\mathrm{B} = 1}$.

f) What value of $B = B_1$ corresponds to the maximal tadpole orbit, i.e., the tadpole orbit which extends to L_3? How close does this orbit get to the secondary? What is its maximum radial width? For $B > B_1$, the orbit is a horseshoe that encircles L_3.

Let's consider the tadpole orbit in the upper half of the plane. L_3 is the turning point, at which $\theta \approx \pi$ and $\dot{\theta} = 0$. The latter relation implies $\Delta = 0$ by (29). Plugging into (32) yields $\boxed{\mathrm{B} = 5/3}$.

The closest approach to m_2 occurs at the other turning point, at which (again) $\dot{\theta} = 0 = \Delta$. Then (32) becomes

$$\begin{aligned}
\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2) = 5 & \\
4\left[\sin^3(\theta/2) - 1\right] &= 5\left[\sin(\theta/2) - 1\right] \\
4\left[\sin(\theta/2) - 1\right]\left[\sin^2(\theta/2) + \sin(\theta/2) + 1\right] &= 5\left[\sin(\theta/2) - 1\right] \\
\Longrightarrow \sin(\theta/2) = \frac{-1 \pm \sqrt{2}}{2} &
\end{aligned} \tag{35}$$

Take the upper root (plus sign) for the minimum θ corresponding to closest approach. The minimum distance d_{min} to m_2 is given by the law of cosines:

$$d_{min}^2 = 1 + (1-\mu)^2 - 2(1-\mu)\cos\theta \tag{36}$$

$$d_{min} \approx \sqrt{2(1-\mu)(1-\cos\theta)} \tag{37}$$

Recall again that $\sqrt{1-\cos\theta} = \sqrt{2}\sin(\theta/2)$. Insert (35) into our expression for d_{min} to find

$$\boxed{d_{min} \approx \sqrt{2} - 1} \tag{38}$$

The maximum radial width occurs at $\theta = \pi/3$, according to part (e). There, $\Delta = \pm\sqrt{8\mu/3}$. Then the total width equals $\boxed{2\sqrt{8\mu/3}}$.

g) What value of $B = B_2$ corresponds to the maximal horseshoe orbit, i.e., the horseshoe orbit that approaches the Hill sphere of the secondary? For these orbits, θ and $2\pi - \theta$

achieve minimum values equal to $F\mu^{1/3}$ where F is a constant of order unity. What is the maximum radial width of these orbits?

Again, at the turning point, $\dot{\theta} = 0 = \Delta$. Insert into (32) to find $\boxed{B \approx 2/(3F\mu^{1/3})}$. The maximal radial width occurs at $\theta = \pi/3$ by (e), and is equal to $\boxed{2\sqrt{8/(3F)}\mu^{1/3}}$.

h) For $B_0 < B < B_1$, calculate the endpoints of small tadpole orbits, i.e., those values of θ where $\dot{\theta} = 0$, for tadpole orbits which never stray far from the Lagrange point about which they librate. Use the relations under (b) and (d). Express the endpoint locations in terms of B.

If $\dot{\theta} = 0$, then $\Delta = 0$. Then (32) reduces to

$$\left[\frac{1}{\sin(\theta/2)} + 4\sin^2(\theta/2)\right] = 3B, \tag{39}$$

This is a cubic equation for $\sin(\theta/2)$. For small tadpole orbits, $\boxed{\theta = \pi/3 + \epsilon}$, where $\epsilon \ll 1$. Then $\sin(\theta/2) \approx (1 - \epsilon^2/8)/2 - \sqrt{3}\epsilon/4$. Insert into (39) and retain terms of order ϵ^2 to find that the left-hand side of (39) equals $3 + 9\epsilon^2/4$. Then $\boxed{\epsilon|_{end} = \pm\sqrt{4(B-1)/3}}$.

i) Use (b), (d), and (h) to derive an expression for the (slow) period of libration of small tadpole orbits. You will need to expand the expression in brackets in (d) about $\theta = \pi/3$ or $\theta = 5\pi/3$. Your expression should not depend on the size of the tadpole in the small tadpole limit. Evaluate this libration period for a Trojan asteroid co-orbiting with Jupiter.

Employ the expansion in (h) to express (32) and (27) as

$$\dot{\theta}^2 + 3\mu(3 + 9\epsilon^2/4) = 9\mu B \tag{40}$$

$$\Longrightarrow \dot{\theta}^2 = 9\mu[(B-1) - \frac{3}{4}\epsilon^2] \tag{41}$$

Now recognize from (h) that $B - 1 = 3\epsilon|^2_{end}/4$. Then we can write

$$\dot{\theta}^2 = \frac{27\mu}{4}[\epsilon|^2_{end} - \epsilon^2] \Longrightarrow \frac{d\theta}{\sqrt{\epsilon|^2_{end} - \epsilon^2}} = \sqrt{\frac{27\mu}{4}}dt \tag{42}$$

But $d\theta = d\epsilon$. The ϵ-integral from $\epsilon = -|\epsilon|_{end}|$ to $\epsilon = |\epsilon|_{end}|$ equals π. The corresponding integral over time equals the $P_{lib}/2$, where P_{lib} is the full libration period. Therefore, $\boxed{P_{lib} = 2\pi\sqrt{\frac{4}{27\mu}}}$. This expression matches that given in Murray & Dermott (3.151). For

Jupiter, $\mu = 10^{-3}$. Remember that in our natural units, 2π is equal to the period of the planet. Jupiter's orbital period is 12 yr. Then the period of slow libration for Trojans on small tadpole orbits is about 144 yr.

Problem 3. *Isolation of Planetary Embryos*

Consider a disk composed of massive planetesimals. The most massive planetesimal has the largest cross-section for accreting other bodies, not only because it has the largest geometric radius but also because it possesses the largest gravitational focussing factor. Thus, the most massive body in the swarm tends to accrete all other bodies in its vicinity. This problem computes the point at which this initial feeding frenzy stops.

A body of mass M at distance r from a star of mass M_* can accrete other bodies within a few Hill radii of its orbit:

$$\Delta r = B(M/3M_*)^{1/3} r$$

where numerical experiments demonstrate that $B \sim 3.5 - 4$ for a quiescent disk of planetesimals (Lissauer 1993, Annual Reviews of Astrophysics, 31, 129). Take σ to be the surface mass density (mass per unit face-on area) of the planetesimal disk. Derive an expression for the "isolation mass," the maximum mass which can accrete within such a disk at every radius. Evaluate the isolation mass in units of an Earth mass for conditions appropriate to the minimum-mass solar nebula: $\sigma \sim 10(r/\mathrm{AU})^{-3/2}\,\mathrm{g/\,cm^2}$, $M_* = M_\odot$.

Take the isolation mass to form from a disk annulus centered at radius r and having total radial width $2\Delta r$. Assume for now that $\Delta r \ll r$ so that the isolation mass equals $2\pi\sigma r \times 2\Delta r$; we can check this later. By definition, the isolation mass has eaten all it can eat; i.e.

$$B\left(\frac{4\pi\sigma r \Delta r}{3M_*}\right)^{1/3} r \approx \Delta r \tag{43}$$

Solve for Δr:

$$\Delta r = B^{3/2} r^2 \left(\frac{4\pi\sigma}{3M_*}\right)^{1/2} \tag{44}$$

Therefore the isolation mass equals

$$\boxed{M_{iso} = (4\pi B\sigma)^{3/2}\, r^3\, (3M_*)^{-1/2}} \tag{45}$$

Plug into the minimum-mass solar nebula and chug out

$$\boxed{M_{iso} = 4 \times 10^{26}\,\mathrm{g}\,(r/\,\mathrm{AU})^{3/4}} \tag{46}$$

or about $0.06 M_\oplus$ at 1 AU, $0.2 M_\oplus$ at 5 AU (Jupiter's location), and $0.7 M_\oplus$ at 30 AU (Neptune's location). For comparison, the mass of the Earth's moon equals $0.01 M_\oplus$. Thus, the initial feeding frenzy is thought to yield bodies of between the Moon's mass and Earth's mass throughout the solar system; this initial phase is referred to as the phase of "runaway growth"; *subsequent* accretion of bodies in the phase of so-called "oligarchic growth" is much slower; so slow, in fact, that it is not clear whether Uranus and Neptune, both of which contain $\sim 10\ M_\oplus$ of solids, could form near their present positions in times shorter than the estimated lifetimes of protoplanetary disks, $\sim 10^7$ yr. See Thommes, Duncan, Levison, & Chambers (1999, Nature, 402, 635) for the most recent proposal of a solution to this problem.