Astro 250: Solutions to Problem Set 3
by Eugene Chiang

Problem 1. *The Titan ringlet and the 1:0 Apsidal Resonance*
The Colombo ringlet, also known informally as the Titan ringlet, is a narrow planetary ring around Saturn that sits within the 1:0 apsidal resonance established by the largest of the Saturnian moons, Titan. This means that the precession rate of the apsidal line of the ringlet matches the mean motion of Titan; Titan appears to pull the ring along.
Denote Titan's mass over Saturn's mass by $M_T/M = 2.366 \times 10^{-4}$*, its semi-major axis by* $a_T = 1.22 \times 10^6\,\mathrm{km}$*, and its mean longitude by* λ_T*. Denote a single ring particle's semi-major axis by* $a = 77871\,\mathrm{km}$*, its eccentricity by* $e = 2.6 \times 10^{-4}$*, and its mean motion by* $\Omega = 2.834 \times 10^{-4}\,\mathrm{rad/s}$*.*
a) Write down, to leading order in e*, the single term of the disturbing function due to Titan (the perturber) that represents the 1:0 apsidal resonance. Leave all variables in symbolic form (do not plug in numbers).*

We are the ring; the perturber is outside. The disturbing function due to an exterior perturber is

$$R' = \frac{GM_T}{a_T}(R_D + \alpha R_E) \tag{1}$$

where $\alpha = a/a_T$, R_D is the direct contribution to the disturbing function, and R_E is the indirect contribution. From the situation at hand, we know the dominant cosine term must be

$$\cos(\tilde{\omega} - \lambda_T) \tag{2}$$

To get the strength of the direct term, look at Murray & Dermott's Table B.4; the relevant term is 4D1.1 where $j = 1$. The leading term is ef_{27}, where f_{27} is given in Table B.7 and equation (B.1). The strength of the indirect term is given by Table B.5; the relevant term is 4E1.2. The leading indirect term is $3e/2$. Putting it all together, we write

$$\begin{aligned}
R' &= \frac{GM_T}{a_T}[ef_{27}\cos(\tilde{\omega} - \lambda_T) + \alpha\frac{3}{2}e\cos(\tilde{\omega} - \lambda_T)] && (3)\\
&= \frac{GM_T}{a_T}e\cos(\tilde{\omega} - \lambda_T)\left\{-\frac{1}{2}\left[2 + \alpha\frac{d}{d\alpha}\right]b_{1/2}^{(1)}(\alpha) + \frac{3}{2}\alpha\right\} && (4)\\
&= -\frac{GM_T}{2a_T}e\cos(\tilde{\omega} - \lambda_T)[2b_{1/2}^{(1)}(\alpha) + \alpha\frac{d}{d\alpha}b_{1/2}^{(1)}(\alpha) - 3\alpha] && (5)\\
&= \boxed{-\tfrac{GM_T}{2a_T}e\cos(\tilde{\omega} - \lambda_T)H_{10}(\alpha)} && (6)
\end{aligned}$$

where we have looked ahead to part (b) of this problem for the definition of H_{10}.
b) Use Lagrange's equations to compute $\dot{a}$*,* $\dot{e}$*, and* $\dot{\tilde{\omega}}$ *for the ring particle. Express in terms of the constant* $\eta = (M_T/M)\Omega\alpha H_{10}/2$*, where* $\alpha = a/a_T$ *and* $H_{10} = 2b_{1/2}^{(1)}(\alpha) + \alpha(d/d\alpha)b_{1/2}^{(1)}(\alpha) - 3\alpha$*.*

Lagrange's equations say $\dot{a} \propto \partial R'/\partial\lambda$. Since R' has no dependence on the mean longitude of the ring particle, $\boxed{\dot{a} = 0}$. To leading order, Lagrange's equation for $\dot{e}$ is

$$\begin{aligned}
\dot{e} &= \frac{-1}{\Omega a^2 e}\frac{\partial R'}{\partial \tilde{\omega}} && (7)\\
&= -\frac{1}{\Omega a^2}\frac{GM_T}{2a_T}H_{10}(\alpha)\sin(\tilde{\omega} - \lambda_T) && (8)
\end{aligned}$$

Apply our usual trick of multiplying numerator and denominator by Ω to find

$$\boxed{\dot{e} = -\tfrac{\Omega M_T H_{10}(\alpha) a}{2 M a_T}\sin(\tilde{\omega} - \lambda_T) = -\eta\sin(\tilde{\omega} - \lambda_T)} \tag{9}$$

Finally, repeat for $\dot{\tilde{\omega}}$:

$$\dot{\tilde{\omega}} = \frac{1}{\Omega a^2 e}\frac{\partial R'}{\partial e} \tag{10}$$

$$\boxed{\dot{\tilde{\omega}} = -\tfrac{\eta}{e}\cos(\tilde{\omega} - \lambda_T)} \tag{11}$$

c) The above expression for $\dot{\tilde{\omega}}$ is incomplete because it only accounts for mean-motion resonant forcing by Titan. What is missing is forcing by the secular potential. Let's just say the complete answer is

$$\dot{\tilde{\omega}} = \langle\text{answer in part b}\rangle + \dot{\tilde{\omega}}_{\text{sec}} \tag{12}$$

where $\dot{\tilde{\omega}}_{\text{sec}} = \dot{\tilde{\omega}}_{\text{Saturn}} + \dot{\tilde{\omega}}_{\text{stuff}}$ is the total additional precession rate induced by the oblateness of Saturn and the secular potential of everything else—nearby rings (remember the Titan ringlet is just 1 narrow ring embedded in Saturn's gigantic ring complex), Titan, other satellites, the Sun, lost pens, etc. In fact, Saturnian oblateness completely overwhelms the other contributions. There is no need to write out explicitly what all these terms are; we will just work with $\dot{\tilde{\omega}}_{\text{sec}}(a)$. (Those of you who did problem 1 of PS 1 know what this function is, but the explicit form is not needed for this problem!)

Equations (9) and (12) are coupled ordinary differential equations. Replace e and ϕ in favor of the variables,

$$h \equiv e\cos\phi \tag{13}$$

$$k \equiv e\sin\phi \tag{14}$$

Write down $\dot{h}$ and $\dot{k}$ in terms of η, ϵ, h, and k.

Well,

$$e = \sqrt{h^2 + k^2} \tag{15}$$

$$\dot{e} = (h\dot{h} + k\dot{k})/e = -\eta \sin\phi \tag{16}$$

Also,

$$\dot{h} = -\dot{e}\cos\phi - e\sin\phi\dot{\phi} \tag{17}$$

$$= -\eta \sin\phi\cos\phi - k\dot{\phi} \tag{18}$$

Now $\dot{\phi} = \dot{\tilde{\omega}} - \Omega_T = -(\eta/e)\cos\phi + \epsilon(a)$. Insert this into (18) to find

$$\dot{h} = \eta \sin\phi\cos\phi - k[-(\eta/e)\cos\phi + \epsilon] \tag{19}$$

$$\boxed{\dot{h} = -\epsilon k} \tag{20}$$

Insert into (16) to find

$$\boxed{\dot{k} = -\eta + \epsilon h} \tag{21}$$

d) Solve your equations for $\dot{h}$ and $\dot{k}$. Your solution should contain two arbitrary constants: an amplitude and a phase associated with a sinusoidal oscillation.

Take another time derivative of (20) and substitute for $\dot{k}$ using (21):

$$\ddot{h} = -\epsilon\eta - \epsilon^2 h \tag{22}$$

The solution of this equation is

$$\boxed{h = e_{free}\cos(\epsilon t + \phi_{free}) + \eta/\epsilon} \tag{23}$$

where e_{free} and ϕ_{free} are constants of integration. Substitute this solution into (21) and solve to find

$$\boxed{k = e_{free}\sin(\epsilon t + \phi_{free})} \tag{24}$$

e) Plot possible trajectories in h and k space. Identify the conditions under which ϕ is circulating or librating. If the particle is librating, what are the libration centers, $\langle\phi\rangle$? What libration centers are associated with $a > a_0$? What centers are associated with $a < a_0$? If you have the correct solution, you should notice that something terrible happens at $a = a_0$. This is simply a deficiency of our low-order theory.

I would make a postscript picture if I had time, but I don't so here goes: The trajectory in h-k space is a circle whose center is located at $(h = \eta/\epsilon, k = 0)$, and whose radius is $|e_{free}|$. $\boxed{\text{If } |e_{free}| > |\eta/\epsilon|, \text{ then the circle encloses the origin and } \phi \text{ circulates.}}$

If $|e_{free}| < |\eta/\epsilon|$, then the circle does not enclose the origin and ϕ librates.

If $a < a_0$, then $\epsilon(a) > 0$ (the secular precession rate increases with decreasing distance from the oblate planet), and the circle is traced out in the clockwise direction at angular speed ϵ. The only possible libration center if $a < a_0$ is $\langle\phi\rangle = 0$.

All the signs are reversed if $a > a_0$, in which case the only possible libration center is $\langle\phi\rangle = \pi$.

Thus, if the ring particle is just outside the resonance ($a > a_0$), then its pericenter is, on average, directed 180° away from Titan, whereas if it is just inside the resonance, then the pericenter, on average, points towards Titan. At exact resonance, our low-order theory explodes (it yields an infinite forced eccentricity, η/ϵ).

Problem 2. *Tilted Rings*

Consider two planets on circular orbits around a star. The inner planet has mass m_1 and semimajor axis a_1, and the outer planet has mass m_2 and semimajor axis a_2. At $t = 0$, the orbit plane of m_2 coincides with the x-y plane; the orbit plane of m_1 is inclined by i_1 and has longitude of ascending node $\Omega_1 = 0$ (on the x-axis); and the two planets happen to be passing conjunction at mean longitude $\lambda_1 = \lambda_2 = 0$ (on the x-axis).

a) The period ratio between the two planets $(a_2/a_1)^{3/2}$ is found to be very well approximated by the integer ratio $k : j$, where k and j are relatively prime (only common factor of the two numbers is 1). It is proposed that the two planets occupy a $k : j$ mean-motion resonance.

How many conjunctions occur before the planets pass conjunction again at $\lambda_1 = \lambda_2 = 0$? Argue from this result that when $k \gg j$, the mean-motion resonance is weak.

Try drawing a few pictures for yourself for low k and j (e.g., 2:1, 3:2, 3:1, 4:1) and you should convince yourself that the number of conjunctions is $|k - j|$. We can show this more rigorously by noting that conjunctions occur when the difference in mean longitudes $\lambda_1 - \lambda_2 = 2\pi i$, where $i = 0, 1, 2, \ldots$. Now $\lambda_1 - \lambda_2 = n_1 t - n_2 t = n_1 t(1 - j/k) = \lambda_1(1 - j/k)$. Set this equal to $2\pi i$ and we find that λ_1 (at conjunction) $= 2\pi k i/(k - j)$ and λ_2 (at conjunction) $= 2\pi j i/(k - j)$. As i cycles from 0, 1, 2, ..., we ask how many unique longitudes are cycled over. Note that when $i = (k - j)$ we have $\lambda_1 = 2\pi k$ and $\lambda_2 = 2\pi j$ which gives the same longitude at conjunction (namely, 0) as $i = 0$; i.e., we have cycled back to the beginning. Thus the number of unique longitudes is given by $i = 0, 1, ..., k - j - 1$, giving a total of $|k - j|$.

b) Secular approximation: Smear m_2 along its orbit so that it becomes a circular wire of uniform linear mass density. Derive to leading order in i_1 the time-averaged disturbing potential felt by m_1 due to m_2. You will need to perform two integrals: one over the wire that is m_2 (integral over θ), and a second over the orbit of m_1 (integral over ψ). See Figure 1. Use Laplace coefficients (see the integral definition on page 237 of MD).

Hint: Write down z in terms of a_1, i_1, and ψ. Also write down d in terms of a_1, a_2, θ, and z. This d is the denominator of the disturbing potential. Write down the integral over θ for the potential of the outer wire as evaluated at a single point on the inner orbit. Taylor expand the *integrand in the small parameter z BEFORE trying to perform the integral over θ. Your θ-integral should produce Laplace coefficients (following the physicist's maxim that any integral we can't do is given an honorary name). Finally time-average (integrate) over ψ.*

You can check your answer by looking up the appropriate secular term (the term that does not depend on any mean longitudes) in Appendix B of MD. Note that s in the notation of MD is actually $\sin i/2$.

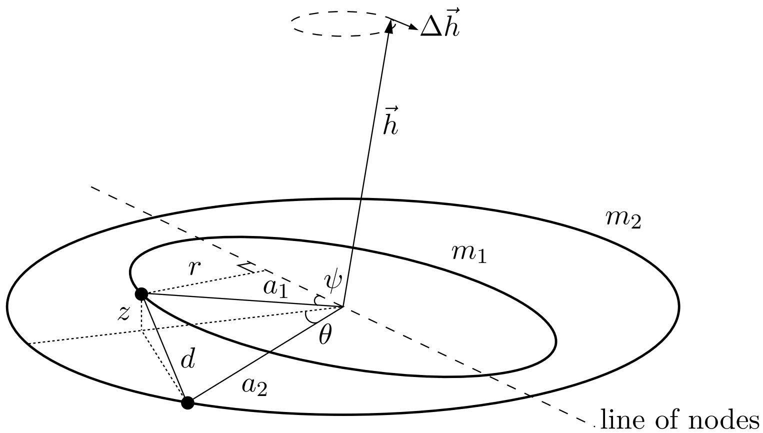

Figure 1: *Schematic of tilted rings problem. The angle θ is in the x-y plane (in the orbital plane of m_2). The angle ψ is the orbital plane of m_1. Note that as shown here in this diagram, the two planets are not actually passing conjunction.*

From the figure,

$$z = a_1 \sin\psi \sin i \tag{25}$$

where ψ measures the angle in the plane of orbit-1 measured from one of the nodes (it doesn't matter whether it's the ascending node or descending node because this is a secular calculation where the objects are smeared into wires with no directions).

Also from the figure,

$$d^2 = z^2 + \ell^2 \tag{26}$$

$$= z^2 + (a_1^2 - z^2) + a_2^2 - 2a_2\sqrt{a_1^2 - z^2}\cos\theta \tag{27}$$

where ℓ is the dotted length in the plane of orbit-2 (not labelled), and for the second line we have applied the law of cosines. (Aside: we will actually never use the length labelled r in the figure; sorry to mislead).

The potential of the entire outer wire evaluated at a single point at ψ on orbit-1 is given by

$$\Phi(\psi) = \int \frac{G\rho(a_2 d\theta)}{d} \tag{28}$$

where $\rho = m_2/(2\pi a_2)$ is the linear mass density of wire-2. Putting everything we have so far together yields

$$\Phi(\psi) = \frac{Gm_2}{2\pi} \int \frac{d\theta}{\left(a_1^2 + a_2^2 - 2a_2\sqrt{a_1^2 - z^2}\cos\theta\right)^{1/2}} \tag{29}$$

Now $\sqrt{a_1 - z^2} \approx a_1(1 - z^2/(2a_1^2))$. Insert into the integral, and bring a_2 out of the parens:

$$\Phi(\psi) = \frac{Gm_2}{2\pi} \int \frac{d\theta}{a_2\left(1 + \alpha^2 - 2\alpha\cos\theta(1 - z^2/(2a_1^2))\right)^{1/2}} \tag{30}$$

where $\alpha \equiv a_1/a_2$. Let's define $\epsilon \equiv z^2/(2a_1^2)$ and $x \equiv 1 + \alpha^2 - 2\alpha\cos\theta$. Then the parens in the integrand equals $(x + 2\alpha\epsilon\cos\theta)^{-1/2} \approx x^{-1/2}(1 - \alpha\epsilon\cos\theta/x)$ and

$$\Phi(\psi) = \frac{Gm_2}{a_2}\frac{1}{2\pi} \int \left(\frac{1}{x^{1/2}} - \frac{\alpha\epsilon\cos\theta}{x^{3/2}}\right) d\theta \tag{31}$$

$$\Phi(\psi) = \frac{Gm_2}{a_2}\left(\frac{1}{2}b_{1/2}^{(0)} - \frac{1}{2}\epsilon\alpha b_{3/2}^{(1)}\right) \tag{32}$$

Now we insert (25) into ϵ:

$$\Phi(\psi) = \frac{Gm_2}{2a_2}\left(b_{1/2}^{(0)} - \frac{\alpha z^2 b_{3/2}^{(1)}}{2a_1^2}\right) \tag{33}$$

$$\Phi(\psi) = \frac{Gm_2}{2a_2}\left(b_{1/2}^{(0)} - \frac{\alpha\sin^2 i_1 b_{3/2}^{(1)}\sin^2\psi}{2}\right) \tag{34}$$

Now we time-average over the orbit of 1. This amounts to a simple angle-average over ψ because orbit-1 is circular. So we recognize that $\int \sin^2\psi d\psi / \int d\psi = 1/2$. And so we have

$$\langle\Phi\rangle = \frac{Gm_2}{a_2}\left(\frac{1}{2}b_{1/2}^{(0)} - \alpha b_{3/2}^{(1)}\sin^2 i_1/8\right) \tag{35}$$

This matches the answer in Appendix B after we realize that in MD's notation, $s \equiv \sin(i/2)$.

Problem 3. *The Disturbing Function Referenced to Inertial Coordinates*
This problem is derived from Goldreich & Tremaine's (1980, ApJ, 241, 425, hereafter GT) landmark treatise on disk-satellite interactions. This paper lays the foundation for understanding planetary ring shepherding and interactions between planets and circumstellar disks.

In a coordinate system that attaches the origin to the (primary) star of mass M, the perturbation potential due to a (secondary) planet of mass M_p reads

$$\phi^p(r,\theta,t) = -\frac{GM_p}{|\vec{r}-\vec{r}_p|} + \frac{GM_p}{|r_p|^3}\,\vec{r}_p\cdot\vec{r}$$

where $\vec{r}$ is the vector position (measured from the origin) where the potential is to be evaluated, and $\vec{r}_p$ is the vector displacement from the origin to the planet. Note that in equation (4) of GT, there is an error; their $(M_s/M_p)\Omega^2(r)$ should be replaced by GM_s/r_s^3. (This error is not propagated throughout the remainder of their paper.)
It is useful to expand ϕ^p in a Fourier series:

$$\phi^p(r,\theta,t) = \sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty}\phi^p_{l,m}(r)\cos\{m\theta - [m\Omega_p + (l-m)\kappa_p]t\}$$

where Ω_p is the mean angular frequency of the planet (the rotational frequency of the guiding center of the planet's orbit), and κ_p is the planet's epicyclic frequency (the frequency of radial oscillations due to non-zero eccentricity of the planet). In a frame that rotates at angular frequency $\Omega_p + (l-m)\kappa_p/m$, the perturbation potential is time-independent and has an m-fold azimuthal symmetry.
Assume that the planet's eccentricity is zero so that $|r_p|$ is a constant. Evaluate the strength of the "principal m^{th} component" of the potential, $\phi^p_{m,m}(r)$. This expression is sufficient to describe the perturbation potential of a planet on a perfectly circular orbit, and it is the component that establishes "principal Lindblad resonances" (Galacto-speak) or "first-order mean-motion resonances" (planeto-speak) in the disk. Principal Lindblad resonances excited in a disk dominate the evolution of the semi-major axis of the planet; they are responsible for planet migration. Express your answer in terms of Laplace coefficients (see the integral definition on page 237 of Murray and Dermott). Watch out for $m=0$ and $m=1$. Compare your answer to equation (7) of GT.

If the planet has no eccentricity, the problem is relatively easy: in the frame rotating with the planet, the planet is fixed. Let's go into this rotating frame, and fix the planet to be on the x-axis in this frame. Define a to be the semi-major axis of the planet, so that the planet is located at $x=a$. Take ψ to be the angle between the x-axis and the position vector, $\vec{r}$, where we wish to evaluate the potential. By the law of cosines, $|\vec{r}-\vec{r}_p|^2 = r^2 + a^2 - 2ar\cos\psi$. And $\vec{r}_p\cdot\vec{r} = ar\cos\psi$. Then we can re-write the potential in terms of coordinates in this rotating frame as

$$\phi^p(r,\theta,t) = -\frac{GM_p}{|\vec{r}-\vec{r}_p|} + \frac{GM_p}{|r_p|^3}\,\vec{r}_p\cdot\vec{r} \tag{36}$$

$$= -\frac{GM_p}{(r^2+a^2-2ar\cos\psi)^{1/2}} + \frac{GM_p}{a^3}ar\cos\psi \tag{37}$$

Now let's examine the Fourier series. The problem asks us to examine $l=m$. Then

$$\phi^p(r,\theta,t) = \sum_{m=0}^{\infty} \phi^p_{m,m}(r)\cos\{m[\theta - \Omega_p t]\} \tag{38}$$

Recognize that $\psi = \theta - \Omega_p t$ (θ is the position angle in inertial space.) To get the Fourier amplitude, $\phi^p_{m,m}(r)$, multiply the above by $\cos m\psi$ and integrate over ψ from 0 to 2π:

$$\int_0^{2\pi} \phi^p(r,\psi)\cos m\psi \, d\psi = \int_0^{2\pi} \phi^p_{m,m}(r)\cos^2 m\psi \, d\psi \quad = \quad \pi\phi^p_{m,m} \text{ (if m} \neq 0) \tag{39}$$

$$= \quad 2\pi\phi^p_{m,m} \text{ (if m} = 0) \tag{40}$$

where we have recalled the wonderful orthogonality property that $\int_0^{2\pi} \cos m\psi \cos n\psi \, d\psi = 0$ if $m \neq n$. Apply the same operation to (37): multiply (37) by $\cos m\psi$ and integrate over a full cycle of ψ:

$$(1+\delta_{m,0})\pi\phi^p_{m,m} = -GM_p \int_0^{2\pi} \frac{\cos m\psi}{(r^2 + a^2 - 2ar\cos\psi)^{1/2}} d\psi + \frac{GM_p ar}{a^3} \int_0^{2\pi} \cos\psi \cos m\psi \, d\psi \tag{41}$$

$$= -\frac{GM_p}{a} \int_0^{2\pi} \frac{\cos m\psi}{(\beta^2 + 1 - 2\beta\cos\psi)^{1/2}} d\psi + \frac{GM_p r}{a^2} \int_0^{2\pi} \cos\psi \cos m\psi \, d\psi \tag{42}$$

where $\beta = r/a$. If $m \neq 1$, then the second integral vanishes. For the first integral, we adopt the physicist's convention that for any integral we can't do, we bequeath it a special name. Let's call it a "Laplace coefficient."

$$b^{(m)}_{1/2} \equiv \frac{1}{\pi} \int_0^{2\pi} \frac{\cos m\psi \, d\psi}{(1 - 2\beta\cos\psi + \beta^2)^{1/2}} \tag{43}$$

If $m = 1$, then the second integral in (42) equals

$$\frac{GM_p r}{a^2} \int_0^{2\pi} \cos\psi \cos m\psi \, d\psi \quad = \quad \frac{GM_p r}{a^2}\pi \tag{44}$$

$$= \quad \frac{GM_p}{a}\pi\beta \tag{45}$$

Putting it all together,

$$\boxed{\phi^p_{m,m} = -\frac{GM_p}{a}\left\{b^{(m)}_{1/2}(1+\delta_{m,0})^{-1} - \delta_{m,1}\beta\right\}} \tag{46}$$

Congratulations: you have derived equation (7) of Goldreich & Tremaine (1980) (Only 103 equations left to go). GT go on to derive expressions for $\phi^p_{m\pm1,m}$, the strengths of leading-order perturbations due to the finite eccentricity of the perturber; these will give rise to so-called "first-order Lindblad resonances" and "first-order co-rotation resonances;" here "first-order" is first-order in the eccentricity of the perturber.

Notice that in $\phi^p_{m,m}$, the indirect term only kicks in for $m = 1$, and it serves to weaken the potential.

Problem 4. *Poincare, Lagrange, and Hamilton*

Hamilton's equations read:

$$\dot{p}_i = \partial H/\partial q_i \tag{47}$$

$$\dot{q}_i = -\partial H/\partial p_i \tag{48}$$

Any set of variables $\{q_i, p_i\}$ that satisfy Hamilton's equations are called "canonical."

Unfortunately, the Keplerian osculating elements are not canonical variables. However, appropriately constructed combinations of the Kepler elements are canonical. One such combination is Poincare's set:

$$q_1 = \lambda \qquad p_1 = \sqrt{\mu a} \tag{49}$$

$$q_2 = -\tilde{\omega} \qquad p_2 = \sqrt{\mu a}\left(1 - \sqrt{1-e^2}\right) \tag{50}$$

$$q_3 = -\Omega \qquad p_3 = \sqrt{\mu a(1-e^2)}\,(1-\cos i) \tag{51}$$

(See section 2.10 of MD but note that they add an extra μ^* into their equations for which we have no use.)

Insert Poincare's canonical variables as written above into Hamilton's equations to derive Lagrange's equations (6.145), (6.146), and (6.148)–(6.150). Ignore (6.147) for which we will have no use in this course. Everywhere you see ϵ in (6.145)–(6.150) replace it with λ (see discussion on page 252).

This is more-or-less a plug-and-chug problem. Historically this is not the way Lagrange actually derived his equations. But the problem does highlight the fact that Lagrange's equations are really just Hamilton's equations, re-written in a nice practical way for celestial mechanicians.

Hint: $\partial R/\partial e = \sum_{i=1}^{3}(\partial R/\partial p_i)(\partial p_i/\partial e)$, and similarly for the other Kepler elements.

Note first that Hamilton's equations as written above utilize the "celestial mechanician's Hamiltonian", i.e., $H = -H_{\mathrm{physicist's}}$. Thus $H = H_{\mathrm{Kepler}} + R$, where R is the celestial mechanician's disturbing function. (Had we used the physicist's H and celestial mechanician's R, then we would have written $H = H_{\mathrm{Kepler}} - R$.) Furthermore, $H_{\mathrm{Kepler}} = \mu^2/(2p_1^2)$

Start with p_1: $\dot{p}_1 = \mu^{1/2}a^{-1/2}\dot{a}/2 = \partial(H_{\mathrm{Kepler}} + R)/\partial\lambda$, whence $\dot{a} = \frac{2}{na}\partial R/\partial\lambda$. This is (6.145).

Next consider p_2: $\dot{p}_2 = \sqrt{\mu}a^{-1/2}\dot{a}(1 - \sqrt{1-e^2})/2 - \sqrt{\mu a}(1-e^2)^{-1/2}(-2e)\dot{e}/2 = -\partial(H_{\mathrm{Kep}} + R)/\partial\tilde{\omega} = -\partial R/\partial\tilde{\omega}$. Solving for $\dot{e}$ using (6.145) gives (6.146).

The derivation of $\dot{i}$ follows similarly: consider $\dot{p}_3 = -\partial(H_{\mathrm{Kep}} + R)/\partial\Omega = -\partial R/\partial\Omega$ and use (6.145) and (6.146). Finally use the trig identity $(1 - \cos i)/\sin i = \tan(i/2)$ to get (6.150).

To get $\dot{\Omega}$, use the hint: $\partial R/\partial i = (\partial R/\partial p_3)\partial p_3/\partial i$ (the other two terms in the sum involve $\partial p_2/\partial i$ and $\partial p_1/\partial i$ which are both zero). Note that $-\partial(H_{\mathrm{Kep}} + R)/\partial p_3 = -\partial R/\partial p_3 = -\dot{\Omega}$ according to Hamilton's equations, and we get (6.148).

To get $\dot{\tilde{\omega}}$, use the hint: $\partial R/\partial e = (\partial R/\partial p_3)\partial p_3/\partial e + (\partial R/\partial p_2)\partial p_2/\partial e$ (the term involving $\partial p_1/\partial e = 0$). Note $-\partial(H_{\mathrm{Kep}} + R)/\partial p_2 = -\partial R/\partial p_2 = -\dot{\tilde{\omega}}$, use (6.148), and we get (6.149).