Astro 250 – Planetary Dynamics – Problem Set 4

Problem 1 is REQUIRED. Do at least 1 other problem in addition to Problem 1.

Readings: Murray & Dermott Chapter 8: 8.3–8.7, and Agol et al. 2005 (we read this article already) on the libration periods and maximum libration amplitudes of first-order resonances in the high and low-eccentricity limits. I condensed all of this material into lecture on Oct 14, so you can just read your lecture notes.

Problem 1: REQUIRED

Write down 1 question or contribute something related to the journal articles on the Google Doc linked to the class webpage.

Problem 2. An Order-of-Magnitude Understanding of First-Order Resonances

Consider a test particle in a first-order $j : j+1$ resonance established by an interior planet. The interior planet has mass μ and occupies a circular orbit of radius 1, in units where $G = M_{\rm central} = 1$. The test particle has eccentricity e.

(a) At the end of lecture on October 14, we derived, following Agol et al. (2005), the libration period $P_{\rm lib}$ and maximum libration width $\Delta a_{\rm lib}$, in the limit of large eccentricity $e > \mu^{1/3}$. Repeat this derivation, explaining all steps.

(b) Derive $P_{\rm lib}$ and $\Delta a_{\rm lib}$ in the limit of low eccentricity $e < \mu^{1/3}$. Note the results you have derived are not found in Murray & Dermott; apparently MD has the wrong results for the low e limit. Compare your answer to Agol et al. (2005).

For both parts (a) and (b), you will use the Tisserand relation for the encounter problem: $\Delta(x^2) \sim \Delta(e^2)$, where Δ denotes the change due to a single encounter (conjunction), and $x \ll 1$ is the semimajor axis difference between the test particle and the perturber.

Problem 3. Inclination Resonance

In lecture on October 14 (and in Section 8.3 of MD), we understood using simple pictures, kicks at conjunctions, and Gauss's perturbation equations (basically $\dot{a} \propto T$) why first-order resonances are stable equilibria. We can also understand why a first-order resonance for a test particle on an eccentric orbit outside a circular planet has a stable point at apoapse; e.g., for the 3:2 resonance, the resonance angle $\phi = 3\lambda' - 2\lambda - \tilde{\omega}'$ librates about π.

Use similar techniques to understand the stability of the corresponding $(i')^2$ resonance, for which the resonance angle $\phi = 6\lambda' - 4\lambda - 2\Omega'$. Explain using simple pictures, kicks at conjunctions, and Gauss's perturbation equations why an inclination resonance can be stable. About what value does ϕ librate?

Problem 4. N petals, forced eccentricities, and another definition of a resonant width

This problem is relevant for the resonant edges of planetary rings.

The edges of planetary rings are near principal Lindblad resonances of azimuthal wavenumber m established by shepherd satellites. At the exact resonance location,

$$(m \mp 1)n - mn_p \pm \dot{\tilde{\omega}} = 0\,. \tag{1}$$

Here m is a positive integer, n and n_p are the mean motions of a ring (test) particle and of the perturbing shepherd, and $\dot{\tilde{\omega}}$ is the apsidal precession rate of the ring particle. The upper/lower signs correspond to inner/outer Lindblad resonances.

Take the shepherd to be outside the ring. The resonant disturbing function of the shepherd is

$$R_{p,res} = \frac{Gm_p}{a_p} f(\alpha) e \cos\phi \tag{2}$$

$$\phi = (m-1)\lambda - m\lambda_p + \tilde{\omega} \tag{3}$$

where λ's are mean longitudes, e is the eccentricity of the test particle, and $f(\alpha) = f(a/a_p)$ is a dimensionless function of the ratio of semi-major axes of the particle to the perturber. f is often of order unity.

a) Calculate $\dot{\tilde{\omega}}_{res}$ and $\dot{e}_{res}$ from $R_{p,res}$ using Lagrange's planetary equations. (We are neglecting the variation in semi-major axis in this first cut to the problem. We can always compute it later.)

b) It is evident that $\dot{\phi} = (m-1)n - mn_p + \dot{\tilde{\omega}}$. In reality, $\dot{\tilde{\omega}} = \dot{\tilde{\omega}}_{res} + \dot{\tilde{\omega}}_{sec}$. For this problem, we will consider $m \neq 1$ and say that $\dot{\tilde{\omega}}_{sec} \ll \dot{\tilde{\omega}}_{res}$. (Note that we cannot ignore $\dot{\tilde{\omega}}_{sec}$ if $m = 1$; see a problem on a previous problem set on the Titan ringlet.) Many planetary rings have their edges located at $m \sim 10$.

Similarly ignore $\dot{e}_{sec}$.

Define $\epsilon(a) = (m-1)n - mn_p$ to write

$$\dot{\phi} = \epsilon(a) + \dot{\tilde{\omega}}_{res} \tag{4}$$

Now take the particle to be firmly in the resonance with vanishingly small libration amplitude; that is, consider the limit $\dot{e} \to 0$ and $\dot{\phi} \to 0$. What are the equilibrium values for e and ϕ? The value for e that you have deduced is called the "forced eccentricity" (as opposed to the "free eccentricity," which is the amplitude of libration in $(h = e\cos\phi, k = e\sin\phi)$ space; see problem on previous problem set on the Titan ringlet). Remember that $\epsilon(a)$ can be either negative or positive, so you should never get a negative eccentricity.

c) Express the eccentricity e in terms of the distance, $x = a - a_0$, where $(m-1)n(a_0) = mn_p$. Of course, we are considering $x \ll a_0$.

d) In the frame co-rotating with the shepherd (which we take to be moving on a perfectly circular orbit), SKETCH APPROXIMATELY the trajectories of ring particles for a few values of x, both positive and negative. You may find it helpful to think in terms of epicyclic frequency, $\kappa = n - \dot{\tilde{\omega}}$ (the frequency of radial oscillations), and the Doppler-shifted azimuthal frequency, $n - n_p$. The particle will make a certain number of radial oscillations for every azimuthal oscillation.

e) What is the value of $x_{crit} > 0$ for which a trajectory at $x = x_{crit}$ just collides with a trajectory at $x = -x_{crit}$ (i.e., on the flip side of the resonance)? This is an estimate of the "width" of the resonance; it is an estimate of the width of the region near the edge of the planetary ring where perturbations by the shepherd satellite are greatest; within x_{crit} of a_0, the velocity dispersion of ring particles can be substantially greater than the velocity dispersion of ring particles in the remainder of the ring that are well removed from the resonance.