Astro 250 – Planetary Dynamics – Solution Set 4

Problem 1 is REQUIRED. Do at least 1 other problem in addition to Problem 1.

Readings: Murray & Dermott Chapter 8: 8.3–8.7, and Agol et al. 2005 (we read this article already) on the libration periods and maximum libration amplitudes of first-order resonances in the high and low-eccentricity limits. I condensed all of this material into lecture on Oct 14, so you can just read your lecture notes.

Problem 2. An Order-of-Magnitude Understanding of First-Order Resonances

Consider a test particle in a first-order $j : j+1$ resonance established by an interior planet. The interior planet has mass μ and occupies a circular orbit of radius 1, in units where $G = M_{\rm central} = 1$. The test particle has eccentricity e.

(a) At the end of lecture on October 14, we derived, following Agol et al. (2005), the libration period $P_{\rm lib}$ and maximum libration width $\Delta a_{\rm lib}$, in the limit of large eccentricity $e > \mu^{1/3}$. Repeat this derivation, explaining all steps.

I'll use my notation (not Agol's) and define $\varepsilon \ll 1$ using $\Omega : \Omega_p = j + \varepsilon : j+1$, where Ω is the mean motion. We can also say $\Omega/\Omega_p \sim (j+1)/j = (a/a_p)^{3/2} = (1+x)^{3/2} \sim 1 + 3x/2$ from which it follows that the semimajor axis displacement $x \sim 2/(3j)$.

The synodic period $P_{\rm syn}$ between successive conjunctions is $2\pi/|\Omega_p - \Omega| \sim 2\pi/[(3\Omega/2)x] \sim 1/x \sim j$.

The maximum libration width $\Delta a_{\rm lib}$ corresponds to a libration amplitude $\Delta\phi \sim 1$. We want an estimate of the time it takes the line of conjunctions to sweep $\Delta\phi$. The number of conjunctions required (per libration "cycle") is $N_c = \Delta\phi/\Delta\ell$, where $\Delta\ell$ is how much the line of conjunctions sweeps (in inertial space) between conjunctions. We estimate:

$$
\begin{aligned}
\Delta\ell &= \Omega_p P_{\rm syn} \quad \bmod 2\pi & (1)\\
&= \Omega_p \frac{2\pi}{\Omega_p - \Omega} \quad \bmod 2\pi & (2)\\
&= \frac{2\pi}{1-\Omega/\Omega_p} \quad \bmod 2\pi & (3)\\
&= \frac{2\pi}{1-(j+\varepsilon)/(j+1)} \quad \bmod 2\pi & (4)\\
&= \frac{2\pi(j+1)}{(1-\varepsilon)} \quad \bmod 2\pi & (5)\\
&\sim 2\pi(j+1)(1+\varepsilon) \quad \bmod 2\pi & (6)\\
&\sim 2\pi j\varepsilon & (7)
\end{aligned}
$$

So $N_c \sim 1/(j\varepsilon)$. And thus $P_{\rm lib} = N_c P_{\rm syn} \sim 1/\varepsilon$.

Now we need to relate ε to e and μ_p (perturber mass). The period of the test particle changes over a libration cycle from $j+\varepsilon$ to j. Thus the fractional change in the test particle's period is ε/j. But by Kepler's Third Law, this must also equal the fractional change in the test particle's semimajor axis. Thus $\varepsilon/j \sim \Delta x/a \sim \Delta x$.

Now use Tisserand, working in the limit that $\Delta x \ll x$ and $\Delta e \ll e$ (high-eccentricity limit):

$$
\begin{aligned}
\Delta(x^2) &\sim \Delta(e^2) & (8)\\
x\Delta x &\sim e\Delta e & (9)\\
\Delta x &\sim \frac{e\Delta e}{x} & (10)\\
\Delta x &\sim \frac{e[(\mu_p/x^2)\times N_c]}{x} & (11)\\
\Delta x &\sim \frac{e\mu_p j^2}{\varepsilon} & (12)
\end{aligned}
$$

Note that Δe is the full change of the test particle's eccentricity over (a quarter of) the libration cycle. The individual eccentricity kicks add coherently over (a quarter of) the libration cycle.

Set this Δx equal to ε/j to find that $\varepsilon \sim \sqrt{e\mu_p}j^{3/2}$, and finally $\boxed{\Delta x = \max \Delta a = \sqrt{je\mu_p}}$ and $\boxed{P_{\rm lib} \sim 1/\varepsilon \sim 1/(\sqrt{e\mu_p}j^{3/2})}$. These are the familiar "square root laws" for a first-order mean-motion resonance.

(b) Derive $P_{\rm lib}$ and $\Delta a_{\rm lib}$ in the limit of low eccentricity $e < \mu^{1/3}$. Note the results you have derived are not found in Murray & Dermott; apparently MD has the wrong results for the low e limit. Compare your answer to Agol et al. (2005).

For both parts (a) and (b), you will use the Tisserand relation for the encounter problem: $\Delta(x^2) \sim \Delta(e^2)$, where Δ denotes the change due to a single encounter (conjunction), and $x \ll 1$ is the semimajor axis difference between the test particle and the perturber.

So everything is the same as part (a) except now the Tisserand relation gives us:

$$\Delta(x^2) \sim \Delta(e^2) \quad (13)$$

$$x\Delta x \sim (\Delta e)^2 \quad (14)$$

$$\Delta x \sim \frac{(\Delta e)^2}{x} \quad (15)$$

$$\Delta x \sim \frac{[(\mu_p/x^2)\times N_c]^2}{x} \quad (16)$$

$$\Delta x \sim \frac{\mu_p^2 j^3}{\varepsilon^2} \quad (17)$$

Set this Δx equal to ε/j to find that $\varepsilon \sim \mu_p^{2/3} j^{4/3}$, and finally $\boxed{\Delta x = \max \Delta a = \mu_p^{2/3} j^{1/3}}$ and $\boxed{P_{\rm lib} \sim 1/\varepsilon \sim 1/(\mu_p^{2/3} j^{4/3})}$, in accord with Agol et al. (2005).

Problem 3. Inclination Resonance

In lecture on October 14 (and in Section 8.3 of MD), we understood using simple pictures, kicks at conjunctions, and Gauss's perturbation equations (basically $\dot{a} \propto T$) why first-order resonances are stable equilibria. We can also understand why a first-order resonance for a test particle on an eccentric orbit outside a circular planet has a stable point at apoapse; e.g., for the 3:2 resonance, the resonance angle $\phi = 3\lambda' - 2\lambda - \tilde{\omega}'$ librates about π.

Use similar techniques to understand the stability of the corresponding $(i')^2$ resonance, for which the resonance angle $\phi = 6\lambda' - 4\lambda - 2\Omega'$. Explain using simple pictures, kicks at conjunctions, and Gauss's perturbation equations why an inclination resonance can be stable. About what value does ϕ librate?

Suppose conjunction occurred at $\lambda' = \Omega' + \pi/2 - \epsilon$, i.e., just shy of when the particle is at its greatest height above the plane. Then the interaction just before conjunction would dominate the interaction just after conjunction, because the particle-perturber distance is shorter before conjunction than after. Before conjunction, the particle is pulled BACK by the perturber, so $T < 0$ which means $\dot{a} < 0$. So the orbit shrinks; the particle's mean motion increases; and the next conjunction occurs at a greater λ' (the perturber needs more time to catch up). So λ' gets accelerated toward $\Omega' + \pi/2$ (first stable point).

Of course if $\lambda' = \Omega' + \pi/2 - \epsilon$, then there is another conjunction at $\lambda' = \Omega' + 3\pi/2 - \epsilon$. The same dynamics unfolds there: λ' gets accelerated toward $\Omega' + 3\pi/2$ (second stable point).

So the stable point for ϕ should be $\pi + 2\pi j$, where j is an integer. This works because $\phi = 6\lambda' - 4\lambda - 2\Omega'$ represents the longitude of the line of conjunctions. At conjunction, $\lambda' = \lambda$; plugging this into ϕ gives $\phi = 2\lambda' - 2\Omega'$, which if $\phi = \pi + 2\pi j$, means that $\lambda = \Omega' + \pi/2 + \pi j/2$, which is consistent with what we said above for the stable points.

I think, but have not checked, that the perturbing acceleration $T \propto i^2$: one power of i to get the vertical component of the perturbing force, and another power of i to get the in-orbit-plane component of the vertical component. This would explain why inclination resonances are second-order in strength.

Problem 4. N petals, forced eccentricities, and another definition of a resonant width

This problem is relevant for the resonant edges of planetary rings.

The edges of planetary rings are near principal Lindblad resonances of azimuthal wavenumber m established by shepherd satellites. At the exact resonance location,

$$(m \mp 1)n - mn_p \pm \dot{\tilde{\omega}} = 0\,. \tag{18}$$

Here m is a positive integer, n and n_p are the mean motions of a ring (test) particle and of the perturbing shepherd, and $\dot{\tilde{\omega}}$ is the apsidal precession rate of the ring particle. The upper/lower signs correspond to inner/outer Lindblad resonances.

Take the shepherd to be outside the ring. The resonant disturbing function of the shepherd is

$$R_{p,res} = \frac{Gm_p}{a_p} f(\alpha) e \cos\phi \tag{19}$$

$$\phi = (m-1)\lambda - m\lambda_p + \tilde{\omega} \tag{20}$$

where λ's are mean longitudes, e is the eccentricity of the test particle, and $f(\alpha) = f(a/a_p)$ is a dimensionless function of the ratio of semi-major axes of the particle to the perturber. f is often of order unity.

a) Calculate $\dot{\tilde{\omega}}_{res}$ and $\dot{e}_{res}$ from $R_{p,res}$ using Lagrange's planetary equations. (We are neglecting the variation in semi-major axis in this first cut to the problem. We can always compute it later.)

Lagrange's equations to leading order in eccentricity read, $\dot{e} = (-1/na^2e)\partial R/\partial\tilde{\omega}$ and $\dot{\tilde{\omega}} = (+1/na^2e)\partial R/\partial e$. These give

$$\boxed{\dot{e} = \frac{m_p}{m_c}\frac{af(\alpha)}{a_p}n\sin\phi} \tag{21}$$

$$\boxed{\dot{\tilde{\omega}} = \frac{m_p}{m_c}\frac{af(\alpha)}{a_p}\frac{1}{e}n\cos\phi} \tag{22}$$

where m_c is the mass of the central object (planet).

b) It is evident that $\dot{\phi} = (m-1)n - mn_p + \dot{\tilde{\omega}}$. In reality, $\dot{\tilde{\omega}} = \dot{\tilde{\omega}}_{res} + \dot{\tilde{\omega}}_{sec}$. For this problem, we will consider $m \neq 1$ and say that $\dot{\tilde{\omega}}_{sec} \ll \dot{\tilde{\omega}}_{res}$. (Note that we cannot ignore $\dot{\tilde{\omega}}_{sec}$ if $m = 1$; see a problem on a previous problem set on the Titan ringlet.) Many planetary rings have their edges located at $m \sim 10$.

Similarly ignore $\dot{e}_{sec}$.

Define $\epsilon(a) = (m-1)n - mn_p$ to write

$$\dot{\phi} = \epsilon(a) + \dot{\tilde{\omega}}_{res} \tag{23}$$

Now take the particle to be firmly in the resonance with vanishingly small libration amplitude; that is, consider the limit $\dot{e} \to 0$ and $\dot{\phi} \to 0$. What are the equilibrium values for e and ϕ? The value for e that you have deduced is called the "forced eccentricity" (as opposed to the "free eccentricity," which is the amplitude of libration in ($h = e\cos\phi, k = e\sin\phi$) space; see problem on previous problem set on the Titan ringlet). Remember that $\epsilon(a)$ can be either negative or positive, so you should never get a negative eccentricity.

$\dot{e} = 0$ demands $\boxed{\phi = 0, \pi}$. $\dot{\phi} = 0$ demands $\dot{\tilde{\omega}}_{res} = -\epsilon(a)$. Recognize that our answer is part (a) is actually $\dot{\tilde{\omega}}_{res}$. Then use (22) to solve for $\boxed{e = |(m_p af(\alpha)/m_c a_p)(n/\epsilon)|}$, where have not bothered to worry about the sign of ϵ and the sign of $\cos\phi$.

c) Express the eccentricity e in terms of the distance, $x = a - a_0$, where $(m-1)n(a_0) = mn_p$. Of course, we are considering $x \ll a_0$.

Write $n = [mn_p/(m-1)](1 + x/a_0)^{-3/2} \approx [mn_p/(m-1)](1 - 3x/2a_0)$ and insert into part (c) to find

$$\boxed{e = \left|\frac{m_p}{m_c}\sqrt{\frac{a_p}{a_0}}\frac{f(\alpha)}{m}\frac{2a_0}{3x}\right|} \tag{24}$$

d) In the frame co-rotating with the shepherd (which we take to be moving on a perfectly circular orbit), SKETCH APPROXIMATELY the trajectories of ring particles for a few values of x, both positive and negative. You may find it helpful to think in terms of epicyclic frequency, $\kappa = n - \dot{\tilde{\omega}}$ (the frequency of radial oscillations), and the Doppler-shifted azimuthal frequency, $n - n_p$. The particle will make a certain number of radial oscillations for every azimuthal oscillation.

For a resonant particle with zero libration amplitude, $\dot{\phi} = 0 = (m-1)n - mn_p + \dot{\tilde{\omega}}$. The epicyclic frequency $\kappa = n - \dot{\tilde{\omega}} = n + (m-1)n - mn_p = m(n - n_p)$. Then in one synodic period, $2\pi/(n - n_p)$, the particle completes m radial oscillations. The amplitude of the radial oscillation, i.e., the maximum radial deviation from the guiding circle, is ea.

Go into the frame rotating with the mean motion of the shepherd, and have the shepherd's (fixed) position be on the x-axis in this frame. Then the ring particle will trace out an m-petalled pattern in this frame; i.e. a flower with m number of petals. When the ring particle achieves conjunction with the shepherd, whether the particle is at its apoapse or at its periapse depends on whether $a < a_0$ or $a > a_0$. If $a > a_0$, then $\epsilon < 0$ which in turn implies that the libration center $\phi = 0$. If $\phi = 0$, then at conjunction ($\lambda = \lambda_p$), $\lambda = \tilde{\omega}$—the particle is at its periapse. Thus, at $a > a_0$, we orient an m-petalled flower (or an m-toothed gear, if you like) such that a trough lies on the x-axis. If $a < a_0$, we orient the m-petalled flower such that a crest lies on the x-axis. The height of the petals decreases as the distance from exact resonance increases; in other words, as $|x|$ increases, ea decreases.

e) What is the value of $x_{crit} > 0$ for which a trajectory at $x = x_{crit}$ just collides with a trajectory at $x = -x_{crit}$ (i.e., on the flip side of the resonance)? This is an estimate of the "width" of the resonance; it is an estimate of the width of the region near the edge of the planetary ring where perturbations by the shepherd satellite are greatest; within x_{crit} of a_0, the velocity dispersion of ring particles can be substantially greater than the velocity dispersion of ring particles in the remainder of the ring that are well removed from the resonance.

The amplitude of each petal is ea_0; a petal at $x < 0$ just touches the petal at $x > 0$ when $ea_0 = x = x_{crit}$. Insert (24) into this equation to find

$$\boxed{x_{crit} = \sqrt{\frac{m_p}{m_c}\frac{2f(\alpha)}{3m}\left(\frac{a_p}{a_0}\right)^{1/2}}\,a_0} \tag{25}$$

To order of magnitude, $x_{crit} \approx \sqrt{m_p/m_c}\,a_0$. For satellites shepherding the ϵ ring of Uranus, this distance is a few km. Thus, in the ϵ ring which measures ~60 km radially, you should imagine the last few km near either resonant edge being stirred up dramatically by the shepherds.