

## Astro 250: Solutions to Problem Set 5

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### Problem 1. Precessing Planes and the Invariable Plane

Consider a star of mass  $m_c$  orbited by two planets on nearly circular orbits. The mass and semi-major axis of the inner planet are  $m_1$  and  $a_1$ , respectively, and those of the outer planet,  $m_2 = (1/2) \times m_1$  and  $a_2 = 4 \times a_1$ . The mutual inclination between the two orbits is  $i \ll 1$ . This problem explores the inclination and nodal behavior of the two planets.

a) What is the inclination of each planet with respect to the invariable plane of the system? The invariable plane is perpendicular to the total (vector) angular momentum of all planetary orbits. Neglect the contribution of orbital eccentricity to the angular momentum. Call these inclinations  $i_1$  and  $i_2$ .

The masses and semi-major axes of the two planets are so chosen as to make the arithmetic easy; the norm of each planet's angular momentum is the same as the other,  $|\vec{l}_1| = |\vec{l}_2| = m_1 \sqrt{G m_c a_1}$ . Orient the x-y axes in the invariable plane so that the y-axis lies along the line of intersection between the two orbit planes (the nodal line). Then in the x-z plane, the total angular momentum vector lies along the z-axis, while  $\vec{l}_1$  lies (say) in quadrant II and makes an angle with respect to the z-axis of  $i_1 > 0$ , while  $\vec{l}_2$  lies in quadrant I and makes an angle with respect to the z-axis of  $i_2 > 0$ . Remember that the mutual inclination  $i = i_1 + i_2$ . Then the total angular momentum vector is

$$\vec{l}_{tot} = |l_1|(\cos i_1 + \cos i_2)\hat{z} + |l_1|(\sin i_2 - \sin i_1)\hat{x} \quad (1)$$

while

$$\vec{l}_1 = |l_1| \cos i_1 \hat{z} - |l_1| \sin i_1 \hat{x} \quad (2)$$

$$\vec{l}_2 = |l_1| \cos i_2 \hat{z} + |l_1| \sin i_2 \hat{x} \quad (3)$$

From  $\vec{l}_1 \cdot \vec{l}_{tot} = |l_1| |\vec{l}_{tot}| \cos i_1$ , it is straightforward to derive that

$$\sqrt{\frac{1 + \cos i}{2}} = \cos i_1 \quad (4)$$

from which it is clear in the small angle limit that  $i_1 = i/2$ . A similar calculation yields

$i_2 = i/2$ . Thus the invariable plane lies right in between the two orbit planes, as it must since each planet contains as much scalar orbital angular momentum as the other.

b) Take for the rest of this problem your reference plane to be the invariable plane. Use lowest-order secular theory to compute the frequencies of nodal precession of each of the two planets,  $\dot{\Omega}_1$  and  $\dot{\Omega}_2$ . Check that these rates do not have any dependence on  $i_1$  and  $i_2$ . They should depend only on the masses and semi-major axes. Be sure to include the sign.

Equation (7.31) of Murray & Dermott gives the answer: for two planets about a star, there is only 1 non-zero nodal eigenfrequency,  $\dot{\Omega}_1 = \dot{\Omega}_2 = f = B_{11} + B_{22}$ . There can be only 1 non-zero eigenfrequency because inclinations are always mutual; for two planets, there is only 1 mutual inclination. The diagonal matrix elements  $B_{jj}$  are given by equation (7.11). Plugging in numbers gives

$$\dot{\Omega}_1 = \dot{\Omega}_2 = -\frac{n_1}{64} \frac{m_1}{m_c} b_{3/2}^{(1)}(0.25) \quad (5)$$

c) How do  $i_1$ ,  $i_2$ , and  $i$  vary in time?

The inclination and nodal behavior for two planets is given by (7.29). One constraint is that initially,  $|\Omega_1 - \Omega_2| = \pi$ . But this is true for all time because of our answer in (b). From (7.29), we have

$$p_1 = i_1 \sin \Omega_1 = I_{11} \sin(ft + \gamma_1) + I_{12} \sin \gamma_2 \quad (6)$$

$$q_1 = i_1 \cos \Omega_1 = I_{11} \cos(ft + \gamma_1) + I_{12} \cos \gamma_2 \quad (7)$$

$$p_2 = i_2 \sin \Omega_2 = I_{21} \sin(ft + \gamma_1) + I_{22} \sin \gamma_2 \quad (8)$$

$$q_2 = i_2 \cos \Omega_2 = I_{21} \cos(ft + \gamma_1) + I_{22} \cos \gamma_2 \quad (9)$$

We are free to choose the initial orientations of the two planes so that  $\Omega_1(t = 0) = \pi/2$  and  $\Omega_2(t = 0) = 3\pi/2$ . This choice renders the initial  $p_1 = i_1(t = 0)$ ,  $q_1 = 0$ ,  $p_2 = -i_2(t = 0) = -i_1(t = 0)$ ,  $q_2 = 0$ . Then

$$I_{11} \sin \gamma_1 + I_{12} \sin \gamma_2 = i_1(t = 0) \quad (10)$$

$$I_{11} \cos \gamma_1 = -I_{12} \cos \gamma_2 \quad (11)$$

$$I_{21} \sin \gamma_1 + I_{22} \sin \gamma_2 = -i_1(t = 0) \quad (12)$$

$$I_{21} \cos \gamma_1 = -I_{22} \cos \gamma_2 \quad (13)$$

These look like 4 equations in 6 unknowns. But in general they really represent 4 equations in 4 unknowns. Remember that  $I_{11}$  is related to  $I_{21}$  since both are components of

the same eigenvector of non-trivial eigenvalue. The components of the other eigenvector,  $I_{12}$  and  $I_{22}$ , are also related to one another, since they are components of the other eigenmode of zero eigenfrequency. From the definition of the matrix  $B$  in equations (7.11) and (7.12), it is easy to see that the two eigenvectors of  $B$  are  $[1 \ -1]$  and  $[1 \ 1]$ . Thus,  $I_{11} = -I_{21}$  and  $I_{12} = I_{22}$ . Inspection reveals the solution  $\gamma_2 = 0$ ,  $\gamma_1 = \pi/2$ ,  $I_{11} = i_1(t=0)$ ,  $I_{21} = -i_1(t=0)$ , and arbitrary  $I_{12} = I_{22}$ . Then

$$p_1 = i_1(t=0) \cos(ft) \quad (14)$$

$$q_1 = -i_1(t=0) \sin(ft) \quad (15)$$

$$p_2 = -i_1(t=0) \cos(ft) \quad (16)$$

$$q_2 = i_1(t=0) \sin(ft) \quad (17)$$

Therefore  $i_1 = \sqrt{p_1^2 + q_1^2} = i_1(t=0) = i/2$  and  $i_2 = \sqrt{p_2^2 + q_2^2} = i_1(t=0) = i/2$ . Thus, both orbit planes remain at the same inclination with respect to the invariable plane for all time, and their nodes regress at the same rate  $f = \dot{\Omega}_1 < 0$ . The mutual inclination  $i$  remains fixed.

In fact, in the Laplace-Lagrange solution, the mutual inclination in *any* 2-planet system remains fixed for all time (in the limit of small mutual inclination). This is easily seen since  $i_2(t=0) > 0$  could have been anything and we would have derived the same solution except that  $I_{21} = -i_2(t=0)$ .

*d) Place a test particle in the invariable plane on a circular orbit at a semi-major axis of  $a_t = 2 \times a_2$ . Does the inclination of the particle with respect to the invariable plane remain zero? If you laid a disk of test particles in the invariable plane, would it remain there?*

**No**, the particle does not remain in the invariable plane. The particle's inclination vector is the vector sum of the free inclination vector and the forced inclination vector; see Figure 7.3. Since the particle's initial inclination is zero, this means that the length of the free inclination vector equals the length of the forced inclination vector, but that the two vectors are initially anti-parallel. They cannot remain anti-parallel since the free inclination vector sweeps at a frequency,  $B$ , defined by equation (7.57), which is, in general, different from the forced (eigen)frequency,  $f$ . [But there do exist certain semi-major axes for which  $B$  will happen to equal to  $f$ ; those are the locations of linear secular resonances (resonances that obtain under linear secular theory). At these locations, the length of the forced inclination vector explodes.]

The same is true for a disk of test particles that lie initially in the invariable plane. **The disk would begin immediately to warp.** Thus, in this sense, there is nothing invariable about the invariable plane.

**Problem 2.** *The Warp Radius of the Laplacian Plane*

By now we are used to the idea that an oblate planet induces apsidal (and nodal) precession in a satellite's orbit. This problem explores the effect of the parent star on satellite precession rates.

Consider a star-planet-satellite system. Recognize that from the satellite's point-of-view, the parent star appears to revolve around the planet; the star can be considered merely another (very massive) satellite on an exterior orbit about the planet. (Who says that the Sun doesn't revolve around the Earth?)

a) Use secular theory to derive the rate of nodal precession of the satellite induced by the star,  $\dot{\Omega}_c$ . Take the mass of the star to be  $m_c$ , the planet mass to be  $m_p$ , the satellite mass to be zero, the satellite's distance to the planet to be  $a$ , and the planet's distance to the star to be  $r$ . Assume that the satellite's orbit plane about the planet is at low, but non-zero, inclination with respect to the planet's orbit plane about the star.

Consider the star and satellite to orbit the spherical planet. This is just like the two-planet Laplace-Lagrange problem, where the inner (satellite) mass  $m_1$  is vanishingly small, and the outer (stellar) mass  $m_2 = m_c$ .

As with the usual 2-planet case, there is only 1 non-zero eigenfrequency; using (7.11) and (7.31), we have

$$\dot{\Omega}_c = -\frac{n_1}{4} \frac{m_c}{m_p} \left(\frac{a}{r}\right)^2 b_{3/2}^{(1)}(a/r) \quad (18)$$

$$= \boxed{-\frac{3}{4} \frac{m_c}{m_p} \left(\frac{a}{r}\right)^3 \sqrt{\frac{Gm_p}{a^3}}} \quad (19)$$

where for the last line we have used the fact that the Laplace coefficient goes to  $3a/r$  for  $a/r \ll 1$ .

b) Take the leading term of equation (6.250) of Murray & Dermott for the nodal precession rate of the satellite induced by the planet's  $J_2$  (oblateness). Those of you who solved problem 1 of problem set #1 should find the leading term of this expression not surprising. Compare this rate,  $\dot{\Omega}_p$ , to  $\dot{\Omega}_c$  and solve for the critical planetocentric radius,  $a_w$ , at which these rates are equal. This is approximately the radius where the Laplacian plane warps. The Laplacian plane is that plane about which test particles nodally precess. At  $a < a_w$ , the Laplacian plane aligns itself with the planet's equator plane. At  $a > a_w$ , the Laplacian plane aligns itself with the planet's orbital plane.

The leading term of (6.250) is  $-\sqrt{Gm_p/a^3}(3J_2(R_p/a)^2/2) \approx \dot{\Omega}_p$ , where  $R_p$  is the planet's radius. Set this equal to  $\dot{\Omega}_c$  to solve for  $a = a_w$ :

$$a_w = (2R_p^2 r^3 J_2 \frac{m_p}{m_c})^{1/5} \quad (20)$$

c) Is the Earth's Moon inside or outside  $a_w$ ? Repeat for the Saturnian satellites, Mimas and Titan.

For the Earth's Moon,  $r = 1.5 \times 10^{13}$  cm,  $R_p = 6.4 \times 10^8$  cm,  $m_p = 6 \times 10^{27}$  g,  $J_2 = 1.1 \times 10^{-3}$ , and  $m_c = 2 \times 10^{33}$  g. Then  $a_w = 6.2 \times 10^9$  cm  $<$   $a_{Moon} = 3.8 \times 10^{10}$  cm. The Moon's orbit plane nodally precesses about the Earth's orbit plane, mostly. Solar perturbations matter more than the quadrupole field of the Earth insofar as the nodal precession rate goes.

For Saturn,  $r = 1.4 \times 10^{14}$  cm,  $R_p = 6.0 \times 10^9$  cm,  $m_p = 5.7 \times 10^{29}$  g,  $J_2 = 1.5 \times 10^{-2}$ , and  $m_c = 2 \times 10^{33}$  g. Then  $a_w = 5.6 \times 10^{11}$  cm  $>$   $a_{Titan}, a_{Mimas}$ . Thus Mimas and Titan are both securely in the quadrupole field of Saturn. Notice that the irregular satellite Phoebe ( $a_{Phoebe} = 1.3 \times 10^{12}$  cm) is not.

### Problem 3. Ring locking

*Narrow rings encircle Uranus and Saturn that are apsidally locked. That is, for a given ring, the inner elliptical edge of the ring is observed to be nearly perfectly apsidally aligned with the outer elliptical edge of the ring. Apsidal locking is puzzling because planetary oblateness ( $J_2$ ) induces differential apsidal precession across the ring. The inner edge wants to precess faster than the outer edge (because the former sits at a smaller semi-major axis than the latter); the edges would precess into one another on fast ( $\sim 10^2$  yr) timescales; streamlines would cross, and the eccentricities of ring particles would be collisionally damped to zero. But the eccentricities of rings are not zero. How can a given ring maintain apsidal lock and precess about the planet as if it were a rigid body? Those of you who witnessed the Hubble Space Telescope movie of the Uranian epsilon ring in the first class know first-hand that indeed that ring is eccentric and that it rigidly precesses, giving rise to the "pulsing" effect in the movie. This problem takes a first qualitative step towards understanding apsidal locking. [Those of you who are really interested can read Chiang & Goldreich (2000, ApJ) or chapter 7 of my thesis. This problem was a real bear. But it is the happiest research problem I have worked on so far, and despite much sweat and toil over 2 decades by great dynamicists it remains incompletely solved.]*

*Idealize a given elliptical ring by two infinitesimally narrow, massive elliptical wires that are in the same plane and that are perfectly apsidally aligned. Take the eccentricity and semi-major axis of the inner wire to be  $e_1$  and  $a_1$ , respectively, and those of the outer wire to be  $e_2$  and  $a_2$ . Take  $e_1, e_2 \ll 1$  and  $\Delta a \equiv a_2 - a_1 \ll a_1, a_2$ .*

*Put both massive wires around an oblate planet. The mutual gravitational attraction of the wires induces differential precession. Planetary oblateness induces differential*

*precession. In fact, the two effects can exactly balance. Use the form of Gauss’s equation and your knowledge of secular theory to deduce the sign of  $\Delta e \equiv e_2 - e_1$ . In other words, for the wire ringlets to remain apsidally aligned, must the distance between the two ringlets be smallest at apoapse or smallest at periapse?*

Two apse-aligned wires having positive  $\Delta e$  are closer together at periapse than at apoapse—drawing a circle inside an ellipse easily demonstrates this. Suppose  $\Delta e > 0$ . Then the wires tug on each other at periapse more than they do anywhere else in the orbit. One might object that the interaction at apoapse dominates since the wires are denser at apoapse, but we are taking the individual wire eccentricities to be vanishingly small, so that we can neglect this density effect.

The outer wire is tugged radially inwards by the inner wire at periapse. Gauss’s equation tells us that  $\dot{\omega}_2 \propto -R \cos f$ , where  $R$  is the radial perturbative acceleration. For the outer wire,  $R < 0$ , so  $\dot{\omega}_2 > 0$ . Similarly,  $\dot{\omega}_1 < 0$  because the inner wire is tugged radially outwards by the outer wire. Then the differential precession rate  $\dot{\omega}_2 - \dot{\omega}_1 > 0$  due to wire self-gravity. Is this of the opposite sign to the differential precession rate induced by the planet’s quadrupole field? Yes;  $J_2$  induces  $\dot{\omega}_2 - \dot{\omega}_1 < 0$ . Thus, ring self-gravity and the planet’s quadrupole field can exactly balance and produce an apse-aligned ring if  $\Delta e > 0$ ; that is, the radial width of a narrow ring is smallest at periapse and greatest at apoapse.

In fact, all known narrow rings that encircle Saturn and Uranus have positive eccentricity gradients ( $de/da > 0$ ), in accordance with our simple qualitative reasoning. The showcase example is Epsilon ring of Uranus, which is 20 km wide at periapse and 120 km wide at apoapse. Positive eccentricity gradients provided the main observations favoring ring self-gravity as the sole mechanism underlying apsidal alignment. But this “standard self-gravity” model yields ring masses (wire masses) that are 1-2 orders of magnitude too small to account for Voyager spacecraft observations. Incorporating forces due to interparticle collisions into our equilibrium equations can remedy this problem. But folding in collisional stresses comes at the cost of permitting negative  $\Delta e$  equilibrium solutions. Thus, it is still not completely understood why narrow rings all have  $\Delta e > 0$ .