Order-of-Magnitude Problems in Planetary Science

or

Adventures in Estimation for the 2005 Kobe Planetary School

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In the following pages please find a detailed outline for my lecture. Since this is a school and not a conference, my first priority is to teach rather than to present new research results. *Therefore many of the lecture items are actually "problem set questions" that you are strongly encouraged to try on your own, before the lecture.* I promise that you will gain much better understanding by trying the problems first by yourself. If you disagree with an answer that I derive in lecture, please let me know during lecture!

These are *order-of-magnitude* problems. They are meant to be answered in a few lines only! Do not be concerned with factors of order unity. For all problems, derive both a symbolic expression and a numerical estimate. You are encouraged to consult the web and/or literature for hints.

I will be working in cgs (a.k.a. God's) units.

These questions are chosen not only because they have known answers, but because they might point to questions that as yet have no (published) answers. I hope this list of problems inspires you in selecting your own research problem.

Topics

I. Planet Formation

II. Planet-Disk Interaction

III. Debris Disks

IV. Mass Loss from Hot Jupiters

0. Some Symbols Used

M, planet mass
R, planet radius
$\mu = M/M_\odot$, ratio of planet's mass to the Sun's
a, semi-major axis (often used as stellocentric distance also)
MMSN, minimum-mass solar nebula
σ, disk surface density in small solid bodies [$\mathrm{g\,cm^{-2}}$]
Σ, disk surface density in large solid bodies [$\mathrm{g\,cm^{-2}}$]
Σ_T, total disk surface density (gas + dust)
Ω, angular frequency [rad $\mathrm{s^{-1}}$]
ρ_p, internal mass density of a body [$\mathrm{g\,cm^{-3}}$]
ρ, volumetric mass density (e.g., of solar nebula).
s, size (radius) of small planetesimals
V, volume
u, velocity dispersion (rms random velocity) of small bodies
v, velocity dispersion of big bodies
$R_{\mathrm{H}} = (\mu/3)^{1/3}a$, Hill radius of planet
$v_{\mathrm{H}} \equiv \Omega R_{\mathrm{H}}$, Hill velocity of a planet
t_{cool}, timescale to cool (thermal or Kelvin-Helmholtz timescale)
t_{collapse}, timescale to collapse gravitationally
c_s, gas sound speed
T, gas temperature
n, number density
ν, kinematic viscosity [$\mathrm{cm^2\,s^{-1}}$]
h, pressure scale height of a gas disk, equal to c_s/Ω
$\tilde{\mu}$, mean molecular weight [g]
r, radius (distance from center of planet in section IV)
w, velocity of wind

Order-of-magnitude properties of a minimum-mass solar nebula (MMSN) heated only by stellar irradiation (Chiang & Goldreich, ApJ, 1997)

$T \sim 150\,(a/\mathrm{AU})^{-3/7}$ K
$c_s \sim 1\,(a/\mathrm{AU})^{-3/14}$ $\mathrm{km\,s^{-1}}$
$h/a \sim 0.05\,(a/\mathrm{AU})^{2/7}$
$\Sigma_T \sim 1500\,(a/\mathrm{AU})^{-3/2}$ $\mathrm{g\,cm^{-2}}$
Dust-to-gas ratio, by mass $\sim 10^{-2}$

I. Planet Formation

A. Terrestrials / Ice Giants (Gas-Free Bodies)

1. *Estimate the time to accrete the Earth at 1 AU in the minimum-mass solar nebula (MMSN). Neglect gas and gravitational focussing.*

2. *Estimate the time to accrete Neptune at 30 AU in the MMSN. Neglect gas and gravitational focussing.*

3. Small bodies are required to form Neptune (Goldreich, Lithwick, & Sari, ARAA, 2004).

    a. Two-groups approximation: Bimodal size distribution.
    "Big" bodies excite velocity dispersions.
    "Small" bodies damp velocity dispersions by inelastic collisions.
    Usually, big bodies grow by accreting small bodies, not other big bodies.
    Usually, $v < u$.

    b. Dispersion-dominated oligarchy: If $u > v_{\rm H} \equiv \Omega R_{\rm H}$ (Hill velocity), then each big body (oligarch) controls an annulus of radial width u/Ω. Accretion is in "runaway" mode.

    c. Shear-dominated oligarchy: If $u < v_{\rm H}$, then each oligarch controls an annulus of radial width $R_{\rm H}$.

    d. *Estimate the size, s, of small bodies having small enough u that Neptune can just form within 1/10 the age of the Solar System. Solve first for the u required for the gravitational focussing factor to be large enough, and solve next for the s required for such u. Assume dispersion-dominated oligarchy. (Be aware that $u >$ 100 m/s probably causes small bodies to shatter one another, but neglect this possibility for this problem.)*

4. Neptune Trojans might be dispersion-dominated oligarchs. (Chiang & Lithwick, ApJ, 2005).

5. Future work should relax two-groups approximation and solve for full size distribution.

B. Giant Planets (Gas-Liquid Rich)

1. Gravitational instability

   a. Requires Toomre $Q \sim c_s\Omega/(\pi G \Sigma_T) \sim 1$ and $t_{\rm cool}/t_{\rm collapse} < 1$ (Gammie, ApJ, 2001; Rafikov, ApJL, 2005).

   b. *Show that $Q \sim 1$ when the vertical component of disk self-gravity is comparable to the vertical component of stellar tidal gravity, evaluated at height h above the midplane.*

   c. *Estimate Q for the MMSN as a function of $a > 1\,\mathrm{AU}$.*

   d. *At 5 AU, enhance Σ_T above the MMSN value until $Q \sim 1$. Estimate $t_{\rm cool}/t_{\rm collapse}$ for this "maximum-mass" solar nebula at 5 AU.*

   e. Gravitational instability might still operate at large $a > 100\,\mathrm{AU}$.

2. Core nucleation (Mizuno, PTP, 1980; Rafikov, astro-ph/0405507, 2005)

   a. Runaway gas accretion when mass of hydrostatic gas envelope surrounding core exceeds mass of core.

   b. *Estimate the minimum mass, $M_{\rm core}$, for a core to undergo runaway gas accretion inside the MMSN at 5 AU. Assume the gas behaves isothermally at temperature T, and solve for hydrostatic equilibrium above the core's surface. The boundary condition for this problem is at infinite distance from the core: at infinity, ρ and T are given by the MMSN. Define the mass of the envelope, $M_{\rm env}$, to be within the first density scale height of the core's surface, and solve for $M_{\rm core} = M_{\rm env}$.*

   c. *Repeat (b), but assume instead that the gas behaves adiabatically.*

   d. Whether gas behaves isothermally or adiabatically depends on uncertain opacities.

   e. Above calculations neglect possible gap opened by core.

II. Planet-Disk Interaction (Gap Opening)

A. Viscous disks diffuse.

1. *A ring has vertical optical depth τ. Estimate the time, t_{col}, it takes a ring particle to collide with another particle.*

2. *Optically thin particle disks (planetary rings, debris disks) having vertical optical depth $\tau \ll 1$:*
   Derive an analytic expression for the radial spreading time, t_{diff}, of a ring of radial width Δa and velocity dispersion u. This is the time the ring takes to approximately double its width.
   Hint: Ring particles spread by random walking.
   The stepsize in the random walk is of order the size of a particle's epicycle.

3. *Optically thick particle disks having vertical optical depth $\tau \gg 1$:*
   Repeat (2).
   Hint: The stepsize is now less than the size of a particle's epicycle.
   Approximate the velocity dispersion as nearly isotropic (i.e., eccentricities $\sim$ inclinations).

4. *Estimate how long it would take Saturn's B ring to double its width.*

5. *Gas protoplanetary disks are observed to accrete (diffuse).*
   Estimate the value of their dimensionless viscosity $\alpha \equiv \nu/(c_s h)$.
   Here $h = c_s/\Omega$ is the vertical pressure scale height of the disk.

6. Source of viscosity in protoplanetary disks is unknown.

7. Magneto-rotational instability (MRI) gives maximum $\alpha \sim 0.1$.
   Application of MRI to protoplanetary disks is uncertain because electrical conductivities of cold, dusty disks are uncertain.

8. Perhaps purely hydrodynamical transport involving vortex modes (Mukhopadhyay et al., astro-ph/0501468, 2005).

B. Planets Repel Disks (Shepherding)

1. *Circular, restricted, planar, 3-body problem: Consider a close encounter between a planet and a test particle.*
   The test particle moves on an initially circular orbit.
   Before encounter, the difference between semi-major axes of the planet and the particle is x.
   For $a \gg x \gg R_{\mathrm{H}}$, estimate the change, Δe, of the particle's eccentricity due to the encounter.

2. *Estimate the change, Δx, in the particle's semi-major axis due to the encounter.*
   Be sure to specify the sign.
   Use the conservation of the Jacobi constant and your answer for (1).
   By answering this question, you will begin to understand how planets can open gaps (Goldreich & Tremaine, Nature, 1979)

3. Same procedure can be applied to estimate degree of stochasticity ("jitter") of planetary migration in planetesimal disks (Murray-Clay & Chiang 2005, in preparation).

4. Torque formula

5. Gaps result from the balance between viscous diffusion and repulsive planetary torque.

6. Shepherded rings are juxtaposed gaps.

7. Gaps are not clean for Jupiter-mass planets in protoplanetary disks because $x \sim R_{\mathrm{H}}$.

III. Debris Disks

A. Transport

1. Poynting-Robertson (P-R) drag arises from anisotropic re-emission of photons by moving grain (Burns, Lamy, & Soter, Icarus, 1979).

2. *Show that the timescale for a grain's semi-major axis to shrink by a factor of 2 due to P-R drag is approximately the time the grain takes to absorb/scatter its own rest mass energy in stellar photons.*

3. Poynting-Robertson drag decreases both semi-major axis and eccentricity on similar timescales.

4. *Consider a narrow ring of planetesimals which collide to produce dust at a constant rate. Dust spirals inward by P-R drag to fill the inner hole. Show that the steady-state surface density, σ, of dust is constant with distance.*

5. Radiation pressure expels small grains having $\beta \equiv F_{\rm rad}/F_{\rm grav} > 1/2$.

6. β barely, if ever, exceeds 1/2 for late-type (G or later) stars because optical efficiency factors ($Q_{\rm abs}$, $Q_{\rm scat}$) decrease with smaller grain sizes.

7. *A grain is created by a collision occurring 10 AU away from β Pictoris (A star). The grain has $\beta \sim 10$. Estimate how long such a grain takes to reach the Earth (distance 15 pc).*

8. Stellar wind induces both pressure and "corpuscular" drag. If wind is radial, physics is identical to P-R drag and radiation pressure.

9. Diffusion by collisions (see II.A.1–2). Presumes relative velocities are so low that collisions are merely inelastic, not destructive.

10. *Show that optically thin ($\tau \ll 1$) rings increase their width as $\Delta a \propto t^{1/3}$ (Petit & Henon, A&A, 1987).*

B. Destructive cascades

1. Differential size distribution dN/ds

2. "Catastrophic dispersal": when a projectile strikes a target and the target disperses into fragments of which the largest has a mass less than half that of the original target. "Dispersal" (as opposed to "disruption") implies fragments do not re-assemble.

3. Yield strength $Y\,[\mathrm{erg/cm^3}]$: $\frac{1}{2}\frac{m_{\rm proj}m_{\rm targ}}{m_{\rm proj}+m_{\rm targ}}v_{\rm rel}^2 \equiv Y V_{\rm targ}$.

4. For small targets, might expect $Y \propto s^0$ from intermolecular bonds.

5. Experiments suggest $Y \propto s^{-0.5}$ (approximately) for meter-sized targets (Holsapple et al., Asteroids III, 2004).

6. For large targets, self-gravitational compression becomes important. Might expect $Y \propto s^2$. (*Show this.*)

7. Actual stress vs. size scaling for large self-gravitating objects is debated.

8. Catastrophic, quasi-steady-state cascade (Dohnanyi, JGR, 1969): For objects of a given size, as much mass is being ground into that size range (from the catastrophic destruction of larger objects) as is being ground out of that size range (to create yet smaller objects).

9. "Size range" is logarithmic; number in a given size range is $\sim(dN/ds)s$.

10. "Quasi"-steady because endpoints of distribution are changing.

11. *Prove that for $Y \propto s^0$, a catastrophic, quasi-steady-state cascade gives $dN/ds \propto s^{-7/2}$.*

12. Such a distribution places most of the mass in the largest objects and most of the surface area in the smallest objects.

13. *Derive how dN/ds scales with s for $Y \propto s^t$ (Pan & Sari, Icarus, 2005).*

C. The Case of AU Mic

1. Corpuscular drag induced by fierce M star wind outweighs P-R drag (Plavchan et al., ApJ, 2005).

2. *Assume that the star AU Mic loses mass at the rate of* $\dot{M}_* \sim 10^2 \dot{M}_\odot$, *where* $\dot{M}_\odot \sim 2 \times 10^{-14} M_\odot \,\mathrm{yr}^{-1}$ *is the mass loss rate from the Sun. Estimate the timescale for corpuscular drag to shrink the orbit of a micron-sized dust grain at 40 AU.*

3. *The vertical optical depth of the AU Mic disk at optical wavelengths is* $\tau \sim 2 \times 10^{-4}$. *Estimate the rate,* $\dot{M}_d$, *at which the AU Mic disk produces dust, assuming that dust production by collisions between parent bodies balances dust removal by corpuscular drag. Express in Ceres-masses per year.*
Hint: Scale from conditions in the Solar System's asteroid belt, where $\tau \sim 10^{-7}$ *and* $\dot{M}_d \sim 1$ *Ceres per 4 billion years (Chiang & Strubbe 2005, in preparation).*

4. Inner vs. outer disk surface brightness profiles.

5. Slope of inner disk surface brightness profile is consistent with corpuscular drag (Chiang & Strubbe 2005, in preparation).

6. Slope of outer disk surface brightness profile suggests outer disk is composed of secondary grains having $1/2 - \beta \ll 1$, where β includes (dominant) contribution from stellar wind pressure. (Chiang & Strubbe 2005, in preparation; LeCavelier des Etangs et al., A&A, 2000).

IV. Mass Loss from Hot Jupiters

A. Ly α absorption from transiting HD 209458b (Vidal-Madjar et al., Nature, 2003).

1. $\sim$15% absorption consistent with disk of optically thick neutral hydrogen of radius $\sim 4R$.

2. *Estimate the minimum mass loss rate for the planet in neutral hydrogen.*

3. Mass loss is driven by photoionization by stellar UV photons.
   Mass loss rate must be corrected for ionization fraction.

4. Parker solution for a thermal, supersonic wind (Parker, ApJ, 1964).

5. $\dot{M} \sim 4\pi \rho_s w_s r_s^2$, where subscript "s" denotes sonic point.

6. At $r = r_s$, $w_s = c_s$ by definition.
   Sonic point is a critical point.
   To avoid singularity at critical point, $\lambda_s \equiv GM\tilde{\mu}_s/(kT_s r_s) = 2$.

7. Avoid solving energy equation by assuming wind is isothermal.
   Guess $T = T_s = 10^4\,\mathrm{K}$ by analogy with HII regions.
   Then $w_s = c_s \approx 10\,\mathrm{km/s}$, for $\tilde{\mu}_s \approx 0.6 m_H$ (fully ionized).
   And $r_s = GM/2c_s^2 = 3.4 \times 10^{10}\,\mathrm{cm} = 3.4\ R$.

8. To derive ρ_s, relate sonic point ("s") to ionization base ("b").
   Ionization base is where stellar UV photons are absorbed (base of Stromgren slab).
   Bernoulli integral I is conserved along a streamline.
   For isothermal flow, $I = \frac{1}{2}w^2 + c_s^2 \ln \rho - GM/r$.

9. *Use the Bernoulli integral to relate ρ_s to ρ_b.*
   Assume $w_b/c_s \ll 1$ (wind is sluggish at ionization base).
   Your relation should contain only ρ_s, ρ_b, and $\lambda_b = GM\tilde{\mu}_s/kT_s r_b$.

10. *Estimate ρ_b using photoionization equilibrium.*
    Assume that the rate of photoionizations equals the rate of radiative recombinations.
    Assume that the stellar UV flux incident on the planet is $F_{\mathrm{UV}} \sim 10^3\,\mathrm{erg\,cm^{-2}\,s^{-1}}$.

11. By contrast, winds from solar system planets are predominantly neutral.
    Collisional recombination is orders of magnitude more efficient than radiative recombination.

12. Still need r_b.
 Estimate r_b by relating ionization base ("b") to "known" reference radius ("0").
 At reference radius, $r_0 = 1.4 R_J = 10^{10}$ cm, $P_0 = 1$ bar, $T_0 \approx 1300$ K.
 Use hydrostatic equilibrium to connect "b" to "0" and estimate r_0.
 Assume that for $r < r_b$, $T = T_0$ and $\tilde{\mu} = \tilde{\mu}_0 = 2.3 m_H$ (fully molecular),
 but that for $r > r_b$, $T = T_s$ and $\tilde{\mu} = \tilde{\mu}_s$.
 While T and μ vary discontinuously across $r = r_b$,
 take the pressure P to vary continuously to ensure stability.

13. ρ_b, r_b together give ρ_s.
 Put it all together to estimate $\dot{M}$.

B. Unresolved issues (Murray-Clay, Chiang, Murray, & Arons 2005, in preparation)

1. Pressure boundary condition (subsonic breezes vs. supersonic wind).

2. Energy equation must be solved, with uncertain contribution
 from cooling by metals.

3. Metals (OI, CII) observed in wind (Vidal-Madjar et al., ApJ, 2004).

4. Photo-ionization equilibrium suspect.

5. Does $\dot{M}$ scale as $F_{\rm UV}$ or as $F_{\rm UV}^{1/2}$?