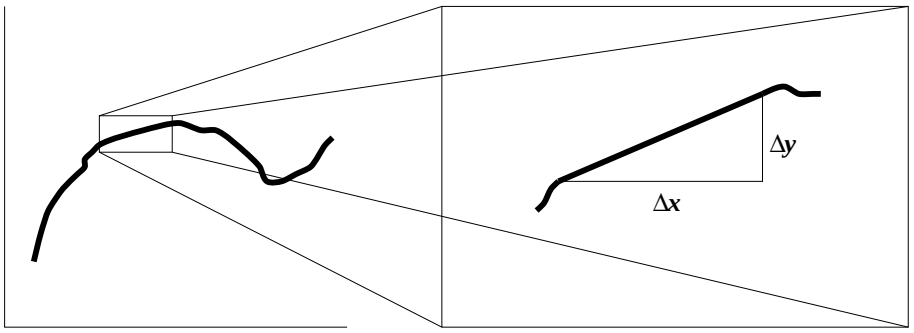
Calculus Made Easy

In this course we will make use of some very simple ideas from calculus. No previous knowledge of calculus is assumed – I will give you all of the formulae that you need and you are not expected to memorize them. For our purposes we will need to be familiar with two concepts, the *derivative* and the *integral*.

Derivative

The derivative of a function is how fast that function is changing at a given point. For example, if I have a function of x, $y(x)$, I can ask how fast that function is changing.



Imagine blowing up a small section of that function. No matter how curly the function looks, if I blow it up enough it will appear straight. Now the slope of the function, it's derivative, is given by $\Delta y/\Delta x$. If I make Δx very small, then we usually write this as dy/dx. We can think of this as d/dx acting on the function $y(x)$. There are numerous rules for taking derivatives of functions. All we will need to know is that for functions $f(x) = x^b$,

$$\frac{d}{dx}x^b = b\,x^{b-1}$$

where b is any real number. Thus for example

$$\frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3}$$

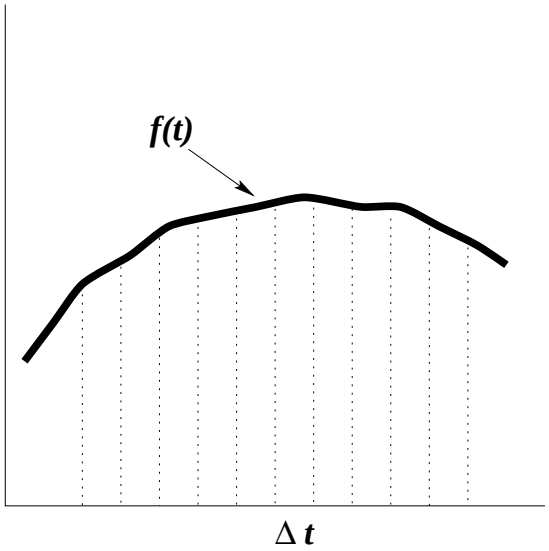
etc. Of course I don't need to have the function be a function of x, it could be a function of time t. For example the scale factor $a(t)$. In this case there exists a shorthand notation for the derivative of $a(t)$ with respect to t, we write

$$\frac{da}{dt} \equiv \dot{a}$$

i.e. we put a dot over the function.

Integral

If the derivative is the limit of small changes in functions, the integral is related to sums or areas. Imagine trying to calculate the area under a curve $f(t)$.



I can break the curve up into infinitely many small segments of width Δt, and on each of these segments the curve is approximately constant. The area of each segment is then $f(t)\Delta t$. If I sum these segments up then I get

$$\text{Sum}\left[f(t)\Delta t\right] = \int f(t)dt$$

where $\int$ is the integral sign (a stretched S for sum). Again we will only need to know how to integrate some very special functions. In particular if $a(t) = t^b$ then

$$\int a(t)dt = \frac{1}{b+1}\, t^{b+1} \qquad \text{if } b \neq -1 \qquad .$$

The case $a(t) = t^{-1}$ is special

$$\int \frac{dt}{t} = \log t$$

Again some examples

$$\int dt = t \qquad \int t^{-2/3}dt = 3t^{1/3} \qquad \int \frac{dt}{\sqrt{t}} = 2t^{1/2}$$

etc.

Finally it is of interest to know how to calculate the area under the curve just for a limited range of t, say from t_1 to t_2. We denote this by

$$\int_{t_1}^{t_2} a(t)dt$$

and the rule is that

$$\int_{t_1}^{t_2} a(t)dt = \left(\int a(t)dt\right)_{\text{at } t=t_2} - \left(\int a(t)dt\right)_{\text{at } t=t_1}$$

so in our previous examples

$$\int_x^y dt = y - x \qquad \int_0^2 t^{-2/3}dt = 3 \times 2^{1/3} \qquad \int_1^x \frac{dt}{\sqrt{t}} = 2\left(x^{1/2} - 1\right)$$

or for the special case of $a(t) = t^{-1}$

$$\int_x^1 \frac{dt}{t} = (\log t)_{t=1} - (\log t)_{t=x} = \log(1) - \log(x) = \log(1/x)$$