Properties of Dirac delta 'functions'

Dirac delta functions aren't really functions, they are "functionals", but this distinction won't bother us for this course. We can safely think of them as the limiting case of certain functions[1] without any adverse consequences.

Intuitively the Dirac δ-function is a very high, very narrowly peaked function with unit area. We may define it by the condition

$$\int dy\ f(y)\delta(x-y) = f(x) \tag{1}$$

for any function $f(y)$. In particular plugging the function $f(y) \equiv 1$ into Eq. (1) shows that the δ-function has unit area. We can write schematically

$$f(x)\delta(x-y) = f(y)\delta(x-y) \tag{2}$$

and

$$\delta(x) = \delta(-x) \tag{3}$$

and

$$\delta(ax) = \frac{1}{a}\delta(x) \quad a > 0 \tag{4}$$

We define derivatives of the δ-function using integration by parts so that

$$\int f(x)\frac{d\delta}{dx}dx = -\int \frac{df}{dx}\delta(x)dx \tag{5}$$

since the surface terms are 0.

One can use these properties to show for example

$$\begin{aligned} \delta(x^2-a^2) &= \delta\left([x-a][x+a]\right) & (6)\\ &= |x-a|^{-1}\delta(x+a) + |x+a|^{-1}\delta(x-a) & (7)\\ &= (2a)^{-1}\left[\delta(x-a)+\delta(x+a)\right] & (8) \end{aligned}$$

The δ-function can be represented as the limit of several common sorts of functions, for example a Gaussian with $\sigma \to 0$ or the limits of $\epsilon/\pi/(x^2+\epsilon^2)$ or $\sin(x/\epsilon)/(\pi x)$ as $\epsilon \to 0$. One representation which will be very useful for us is the Fourier Transform of 1, or

$$\delta(x) \equiv \lim_{T\to\infty}\int_{-T}^{T}\frac{dk}{2\pi}\ e^{ikx} \tag{9}$$

Note that when $x = 0$ the exponential is 1 and the integral is infinite (the function value is very large) while if $x \neq 0$ the integral is highly oscillatory and will evaluate to 0.

[1] See e.g. `http://mathworld.wolfram.com/DeltaFunction.html` for some examples.