Halo bias

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The halo model throughout uses the concept that peaks of the density field cluster differently that the 'average' mass field. The history of 'peaks bias' is complex, but let us simply derive the amount by which we expect peaks to be biased wrt to the underlying mass.

There are a number of derivations of peaks bias of increasing sophistication. Here I present the simplest which is based on a concept known as the peak-background split (see Efstathiou et al. 1988, Cole & Kaiser 1989 and Mo & White 1996 for other derivations). Imagine that there are only 2 wavelengths in our power spectrum, one 'long' wavelength background mode and a shorter wavelength mode. The density contrast of peaks is

$$\delta_P = \frac{N}{\bar{n}V} - 1 \equiv b\delta \tag{1}$$

where N is the number of peaks and $\bar{n}$ is the mean density. If we form a halo/galaxy/object when the density crosses some critical value δ_c being on a swell of the long wavelength mode means we need only fluctuate up by $\delta_c - \delta_{\rm long}$. If I put this threshold in the mass function and expand out

$$\log N = \log N(\delta_c) - \delta \left. \frac{d\log N}{dt} \right|_{t=\delta_c} \tag{2}$$

The first term is the mean density so in Eulerian space, i.e. with volumes fixed, the linear bias term is simply the coefficient of δ or

$$b_E = \left. \frac{d\log N}{dt} \right|_{t=\delta_c} \tag{3}$$

$$= \frac{1}{\delta_c} - \frac{\delta_c}{\sigma^2} \tag{4}$$

$$= \frac{\nu^2 - 1}{\delta_c} \tag{5}$$

with $\nu = \delta_c/\sigma$. Unfortunately we don't live in Eulerian space but rather in Lagrangian space. We go from Eulerian bias to Lagrangian bias by recalling that on our swell the volume has changed by a factor of $1 - \delta$ (to contain the same mass) in the denominator of Eq. (1), thus

$$b(\nu) = 1 + \frac{\nu^2 - 1}{\delta_c} \tag{6}$$

The generalization to other mass functions has been presented by Sheth & Tormen (1999).

This formula provides quite a good fit to the bias of halos in N-body simulations, with bias defined in the sense of $P(k)$, if the halos are mass selected. Note an immediate consequence of this is that the bias of an object of constant mass evolves with time. Also note that if we integrate over some range of masses, the average bias of the objects will be simply

$$\langle b \rangle = \bar{n}^{-1} \int f(\nu) b(\nu) d\nu^2 \tag{7}$$

For a Gaussian field it is also possible to calculate the bias of peaks by explicit computation of the correlation function. For a Gaussian

$$P(\delta_i) \propto (\det C)^{-1/2} \; e^{-\delta^T C^{-1} \delta/2} \tag{8}$$

with $\langle \delta_i \delta_j \rangle = C_{ij}$. If we consider simply two points in space then

$$C = \begin{pmatrix} \xi_0 & \xi_1 \\ \xi_1 & \xi_0 \end{pmatrix} \tag{9}$$

with $\xi_0 = \sigma^2$ and $\xi_1 = \xi(r_{12})$. The probability that both δ_1 and δ_2 are above $\nu\sigma$ is

$$P_2 = \int_{\nu\sigma} \int_{\nu\sigma} d\delta_1 d\delta_2 \; P(\delta_1, \delta_2) \equiv P_1^2 \left(1 + \xi_\nu(r_{12})\right) \tag{10}$$

where

$$P_1 = \int_{\nu\sigma} d\delta \; P(\delta) \to \frac{1}{\sqrt{2\pi}} e^{-\nu^2/2} \tag{11}$$

where the limit is taken for $\nu \to \infty$.

If we take the limit $\xi_1 \ll \xi_0$ and $\nu \gg 1$ then we can simplify

$$P_2 \simeq \frac{1}{2\pi} \int_{\nu\sigma} dt \; e^{-\nu^2/2} e^{A\nu t} = \frac{1}{2\pi} e^{-\nu^2} e^{A\nu^2} \tag{12}$$

where $A \equiv \xi_1/\xi_0 \ll 1$. Comparing to P_1 we see that the bias is ν/σ. The same result can be derived using the peak-background split as

$$P(\delta > \nu\sigma) = P(\delta_s > \nu\sigma - \delta_L) \simeq P \left[1 - \frac{1}{\sigma} \frac{d \log P}{d\nu} \delta_L \right] \tag{13}$$

since P is a Gaussian.