The Press-Schechter Mass Function

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One of the most fundamental predictions of hierarchical structure formation is the mass function: the number (density) of objects as a function of their mass M. Accurate mass functions are used in a number of areas in cosmology; in studies of galaxy formation, in measures of volumes (e.g. galaxy lensing) and in attempts to infer the normalization of the power spectrum, the statistics of the initial density field, the density parameter or the equation-of-state of the dark energy from the abundance of rich clusters. One of the most intriguing aspects of the mass function is that it appears universal, in suitably scaled units, for a wide range of theories. A complete understanding of this phenomenon currently eludes us.

Many years ago Press & Schechter advanced a theory for the mass function which provides a relatively good fit to the results of numerical simulations in CDM-like theories. The P-S mass function agrees relatively well with the results of numerical simulations both for critical density models with power-law spectra and, more surprisingly, for models without this self-similar evolution. The level of agreement is striking because the theoretical underpinnings of Press-Schechter theory and very shaky, and many of the assumptions are known to be wrong!

The P-S mass function and numerical results are known to deviate in detail at both the high and low mass ends. Refinements to this theory have been advanced, all of which relate the abundance of collapsed objects to peaks in the initial density field in a 'universal' manner. The theory can be extended in a number of ways, in some cases improving agreement with numerical results and in some cases making completely false predictions.

The Press-Schechter theory begins with the ansatz that the fraction of mass in halos more massive than M is related to the fraction of the volume in which the smoothed initial density field is above some threshold δ_c. A variety of smoothing windows and thresholds have been advocated, but the most common is a top-hat window in real space and $\delta_c \simeq 1.69$. This value is equal to the extrapolated linear overdensity at which a spherical top-hat perturbation would collapse (see later).

To 'derive' the mass function let us work in k-space, even though during the application we will work in real space. Imagine applying a sequence of sharp low-pass k-space filters to the power spectrum. For a given sequence

$\{k_i\}$ the density field would have dispersion $\{\sigma_i^2\}$ where

$$\frac{d\sigma^2}{d\log k} \equiv \Delta^2(k) = \frac{k^3 P(k)}{2\pi^2} \quad . \tag{1}$$

Since, in linear theory, the k-modes are independent, the density field $\delta(x)$ in real space makes a random walk as k is increased. To $\delta(x)$ we add a random value drawn from a Gaussian of mean 0 and dispersion $d\sigma^2$. This describes a diffusion process where half of the time the density will increase on each step and half of the time it will decrease. So the number of paths at (δ, σ^2) satisfies

$$\frac{dP}{d\sigma^2} = \frac{1}{2}\frac{d^2P}{d\delta^2} \quad . \tag{2}$$

We want to know how many paths reach (δ, σ^2) without ever crossing δ_c at lower σ^2. Solving the diffusion equation by Fourier transform in δ and imposing the condition $P(\delta_c) = 0$ by the method of images (a positive 'source' at $\delta = 0$ and a negative 'source' at $\delta = 2\delta_c$) we obtain

$$P(\delta, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\left[\exp\left(-\frac{\delta^2}{2\sigma^2}\right) - \exp\left(-\frac{(\delta-\delta_c)^2}{2\sigma^2}\right)\right] \quad (\delta < \delta_c) \tag{3}$$

where the normalization is chosen so that $P(\delta, 0)$ integrates to unity.

So the probability for objects which are not collapsed is

$$1 - F(>\delta_c) = \int_{-\infty}^{\infty} d\delta \; P(\delta, \sigma^2) = \mathrm{erf}\left(\frac{\delta_c}{\sqrt{2}\sigma}\right) \tag{4}$$

To find the (comoving) number (density) per unit mass we differentiate $F(>\delta_c)$ wrt M and convert from volume fraction to mass fraction by multiplying by $\bar{\rho}/M$ giving

$$\frac{dn}{dM} = \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M}\frac{\delta_c}{\sigma^2}\frac{d\sigma}{dM}\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right] \tag{5}$$

which is the well-known Press-Schechter mass function. Note that this states that the mass function depends only on the 2-point function of the initial density field, extrapolated to the present using linear theory!

Multiplicity function

In using the Press-Schechter formalism it is useful to make the 'universal' nature more apparent. Under the P-S *ansatz* all of the cosmology dependence is contained within the rms density fluctuation, $\sigma(M)$, smoothed with a top-hat filter on a scale[1] $R^3 = 3M/4\pi\bar{\rho}$. The multiplicity function,

$$\nu\, f(\nu) d\nu \equiv \frac{M}{2\bar{\rho}} \frac{dn}{dM} dM \qquad , \tag{6}$$

is a universal function of the peak height ν which is related to the mass of the halo through

$$\nu \equiv \frac{\delta_c}{\sigma(M)} \tag{7}$$

with $\delta_c = 1.69$. Note that some authors, particularly Sheth & Tormen, define ν to be $(\delta_c/\sigma(M))^2$ rather than $\delta_c/\sigma(M)$ as we have done. The P-S mass function is simply

$$\nu f(\nu) \propto e^{-\nu^2/2} \tag{8}$$

where the normalization constant is fixed by the requirement that all of the mass lie in a given halo

$$\int \nu\, f(\nu) d\nu = \frac{1}{2} \qquad . \tag{9}$$

There is no justification for this normalization from N-body simulations, which cannot probe the $M \to 0$ tail, but we shall adopt it throughout.

In addition to the apparent universality of the multiplicity function, it is easy to derive 'mass weighted' statistics using it. Suppose for example we wanted to compute the mass weighted temperature, or velocity dispersion or potential depth. For a given halo each of these is a function of the halo mass, and we simply sum over the halos by integrating against the multiplicity function, viz:

$$\begin{aligned} \langle (1+\delta) X \rangle &= \int \frac{M}{\bar{\rho}} X(M)\ dn & (10) \\ &= \int X(\nu) f(\nu) d\nu^2 & (11) \end{aligned}$$

[1]In principle R could be defined with respect to $\rho_{\rm crit}$, but this is not the natural choice in the top-hat collapse model.

Spherical top-hat collapse review

The spherical top-hat ansatz describes the formation of a collapsed object by solving for the evolution of a sphere of uniform overdensity δ in a smooth background of density $\bar{\rho}$. By Birkhoff's theorem the overdense region evolves as a positively curved Friedman universe whose expansion rate is initially matched to that of the background. The overdensity at first expands but, because it is overdense, the expansion slows (relative to the background) and eventually halts before the region begins to recollapse. Technically the collapse proceeds to a singularity but it is assumed in a "real" object virialization occurs at twice[2] the turn-around time, resulting in a sphere of half the turn-around radius. In an Einstein-de Sitter model the overdensity (relative to the critical density) at virialization is $\Delta_{\rm c} = 18\pi^2 \simeq 178$. If $\Omega_{\rm m} < 1$ there are two conventions for Δ_c which differ by a factor of $\Omega_{\rm m}$! Some people stick with Δ_c defined wrt the (redshift dependent) critical density while others specify it relative to the background density. The linear theory extrapolation of this overdensity is normally denoted $\delta_{\rm c}$ and is $(3/20)(12\pi)^{2/3} \simeq 1.686$ in an Einstein-de Sitter model. This overdensity is often used as a threshold parameter in PS theory and its extensions and has a very weak cosmology dependence which is often neglected.

[2]There is a small correction to this in the presence of a cosmological constant which contributes a Λr^2 potential.