Cosmological Perturbation Theory

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Outline

Goal: understand large-scale structure

- Baryon acoustic oscillations imprint characteristic scale on matter distribution (Standard Ruler)
- Matter fluctuations amplified by growth function
- Dark energy dominates growth function today (at low z)
- Therefore measuring matter distribution today tells us about dark energy!

Why perturbation theory

- Need to run large number of N-body simulations to compute statistical observables (e.g. power spectrum)
- BAO scale is large (~ 100 Mpc/h), so need to run large volume simulations
- Simulations are expensive!
- Analytic solution computes statistical quantities directly
- Direct analytic solution impossible (non-linear equations of motion), so must resort to perturbation theory

Unperturbed cosmology

- Start well after matter-radiation equality
- Flat FRW cosmology with Λ, ignore radiation and neutrinos
- Friedmann equation: $H^2 = \frac{8\pi G}{3}\bar{\rho} + \frac{\Lambda}{3}$
- Mean density: $\bar{\rho} \propto a^{-3}$
- Later will restrict attention to Einstein-de Sitter cosmology: $\Omega_m = 1,\ \Lambda = 0$

Matter fluid

- Newtonian gravity (distance scales well within the horizon)
- Non-relativistic fluid
- Pressureless, collisionless, zero viscosity
- Assumptions good for cold dark matter
- Assumptions fail for baryons, but only in regions of high density

Peculiar velocity field

- Single-stream approximation (no shell crossing)
- Irrotational: vorticity $\mathbf{w} \equiv \nabla \times \mathbf{v} = 0$
- True in linear theory: $\mathbf{w}$ decays as a^{-1}
- (Not clear when/where these assumptions break down, but the show must go on)

Cosmological coordinates

- Comoving coordinates: $\mathbf{x} = \mathbf{r}/a$
- Conformal time: $\tau = \int dt/a$ or $d\tau = dt/a$
- Metric: $ds^2 = a^2(\tau)[-d\tau^2 + d\mathbf{x}^2]$

Equations of motion for a single particle

- Non-relativistic action:

$$S = \int dt \left[\frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 - m\Phi \right]$$

$$= \int d\tau \, a(\tau) \left[\frac{1}{2} m \left(\frac{d\mathbf{x}}{d\tau} \right)^2 - m\Phi \right]$$

$$\Phi(\mathbf{x}, \tau) = a^2(\tau) \int d^3x' \, \frac{\delta\rho(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|}$$

- Equations of motion:

$$\frac{d\mathbf{x}}{d\tau} = \frac{\mathbf{p}}{ma}, \qquad \frac{d\mathbf{p}}{d\tau} = -ma\nabla\Phi$$

Phase space distribution function

- $dN = f(\mathbf{x}, \mathbf{p}, \tau)\, d^3x\, d^3p$
- For a collection of point masses,

$$f(\mathbf{x}, \mathbf{p}, \tau) = \sum_\alpha \delta^3(\mathbf{x} - \mathbf{x}_\alpha(\tau))\, \delta^3(\mathbf{p} - \mathbf{p}_\alpha(\tau))$$

- Mass density: $\rho(\mathbf{x}, \tau) = m a^{-3}(\tau) \int f(\mathbf{x}, \mathbf{p}, \tau)\, d^3p$
- Momentum density:
  $\rho(\mathbf{x}, \tau)\mathbf{v}(\mathbf{x}, \tau) = a^{-4}(\tau) \int f(\mathbf{x}, \mathbf{p}, \tau)\, \mathbf{p}\, d^3p$
- All higher moments of f are products of ρ and $\mathbf{v}$

Collisionless Boltzmann equation

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{\mathbf{p}}{ma} \cdot \nabla f - ma\nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

- Conservation of phase space volume
- Taking moments gives fluid equations

Fluid equations

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot \mathbf{v} = -\nabla \cdot (\delta \mathbf{v}) \qquad \text{(Continuity)}$$

$$\frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H}\mathbf{v} + \nabla \Phi = -(\mathbf{v} \cdot \nabla)\mathbf{v} \qquad \text{(Euler)}$$

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \delta = \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) \delta \qquad \text{(Poisson)}$$

- $\mathcal{H} = d \ln a / d\tau = aH$
- $\rho(\mathbf{x}, \tau) = \bar{\rho}(\tau)[1 + \delta(\mathbf{x}, \tau)]$
- $\mathbf{v}$ = peculiar velocity ($\mathbf{v} = 0$ at zeroth order)

Linearized fluid equations

- Assume δ and $\mathbf{v}$ small, of the same order
- Drop right-hand sides of fluid equations:

$$\frac{\partial \delta}{\partial \tau} + \theta = 0$$

$$\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + \frac{3}{2}\Omega_m(\tau)\mathcal{H}^2\delta = 0$$

$$\Longrightarrow \boxed{\frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta}{\partial \tau} - \frac{3}{2}\Omega_m(\tau)\mathcal{H}^2\delta = 0}$$

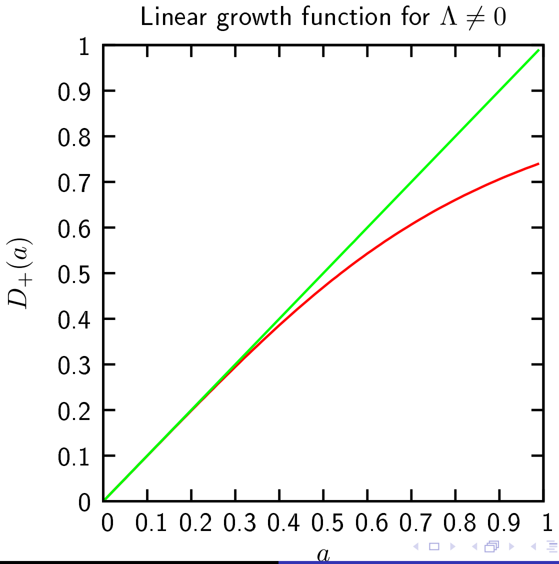
- $\theta \equiv \nabla \cdot \mathbf{v}$ (peculiar velocity divergence)

Growth function

$$\frac{d^2D}{d\tau^2} + \mathcal{H}(\tau)\frac{dD}{d\tau} - \frac{3}{2}\Omega_m(\tau)\mathcal{H}^2(\tau)D = 0$$

- Two linearly indepedent solutions: $D_+(\tau)$ (growing) and $D_-(\tau)$ (decaying)
- Ignore decaying solution: $\delta_L(\mathbf{x},\tau) = D_+(\tau)\delta_0(\mathbf{x})$
- For Einstein-de Sitter universe (or during matter domination), $D_+ \propto a$ and $D_- \propto a^{-3/2}$
- When $\Lambda \neq 0$, D_+ falls below a at late times

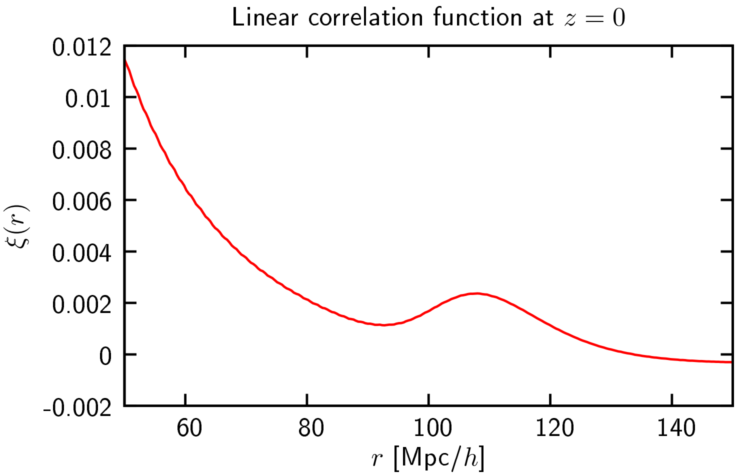
Growth function plot



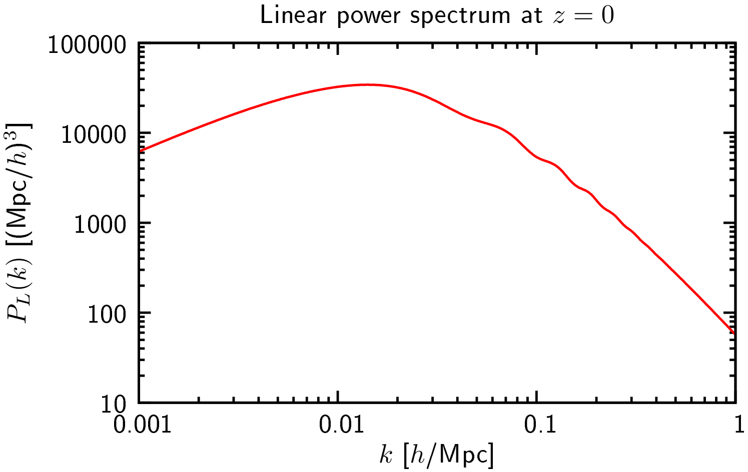
Statistical observables

- Correlation function: $\langle \delta(\mathbf{x})\delta(\mathbf{x}') \rangle = \xi(|\mathbf{x} - \mathbf{x}'|)$
- Baryon acoustic peak at $r \approx 105$ Mpc/h
- Power spectrum: $\langle \tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{k}') \rangle = P(k)\delta^3(\mathbf{k} + \mathbf{k}')$
- $P(k)$ is just the Fourier transform of $\xi(r)$
- At linear order $P_L(k, \tau) = D^2(\tau) P_0(k)$

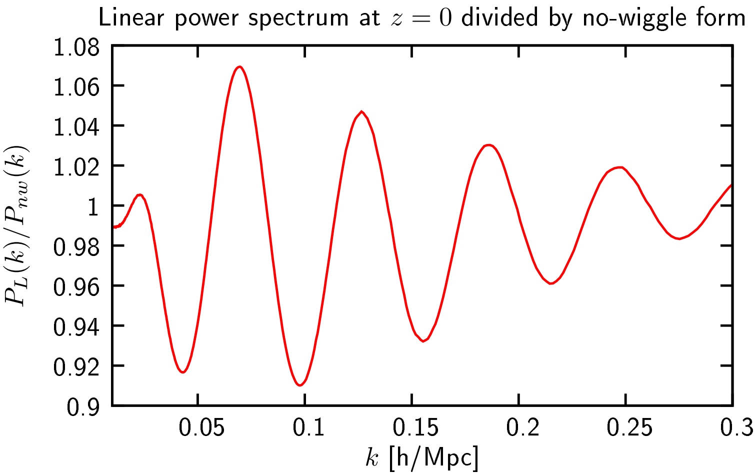
Correlation function



Power spectrum



Power spectrum



Standard Perturbation Pheory

- Basic theory worked out long ago [reviewed in Peebles 1980]
- Explicit formulas and diagrammatic methods developed in 80's and 90's [Fry 1984, Goroff et al 1986, Makino et al 1992]
- Basis for most other perturbative theories

Fluid equations in Fourier space

- Velocity field: $\tilde{\mathbf{v}}(\mathbf{k}) = -\frac{i\mathbf{k}}{k^2}\,\tilde{\theta}(\mathbf{k})$
- RHS of continuity equation:

$$\mathrm{FT}[-\nabla\cdot(\delta\mathbf{v})]$$
$$= -i\mathbf{k}\cdot\int d^3q_1\,d^3q_2\,\delta^3(\mathbf{q}_1+\mathbf{q}_2-\mathbf{k})\frac{-i\mathbf{q}_1}{q_1^2}\tilde{\theta}(\mathbf{q}_1)\tilde{\delta}(\mathbf{q}_2)$$

- RHS of Euler equation (after taking divergence):

$$\mathrm{FT}[-\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]] = -i\mathbf{k}\cdot\int d^3q_1\,d^3q_2\,\delta^3(\mathbf{q}_1+\mathbf{q}_2-\mathbf{k})$$
$$\times\left(\frac{-i\mathbf{q}_1}{q_1^1}\cdot i\mathbf{q}_2\right)\frac{-i\mathbf{q}_2}{q_2^2}\,\tilde{\theta}(\mathbf{q}_1)\,\tilde{\theta}(\mathbf{q}_2)$$

Fluid equations in Fourier space

$$\frac{\partial \tilde{\delta}}{\partial \tau} + \tilde{\theta} = -\int d^3q_1\, d^3q_2\, \delta^3(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{k}) \frac{\mathbf{k}\cdot\mathbf{q}_1}{q_1^2} \tilde{\theta}(\mathbf{q}_1)\tilde{\delta}(\mathbf{q}_2),$$

$$\frac{\partial \tilde{\theta}}{\partial \tau} + \mathcal{H}\tilde{\theta} + \frac{3}{2}\Omega_m \mathcal{H}^2 \tilde{\delta}$$

$$= -\int d^3q_1\, d^3q_2\, \delta^3(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{k}) \frac{k^2(\mathbf{q}_1\cdot\mathbf{q}_2)}{2q_1^2 q_2^2}\, \tilde{\theta}(\mathbf{q}_1)\, \tilde{\theta}(\mathbf{q}_2).$$

- Non-linearity manifested as convolution in Fourier space (mode-coupling)

Perturbation expansion

$$\tilde{\delta}(\mathbf{k},\tau) = \sum_{n=1}^{\infty} \tilde{\delta}^{(n)}(\mathbf{k},\tau), \quad \tilde{\theta}(\mathbf{k},\tau) = \sum_{n=1}^{\infty} \tilde{\theta}^{(n)}(\mathbf{k},\tau).$$

- Insert perturbation expansion in fluid equations, solve order by order
- Simplification for Einstein-de Sitter universe:

$$\tilde{\delta}^{(n)}(\mathbf{k},\tau) = a^n(\tau)\delta_n(\mathbf{k}), \quad \tilde{\theta}^{(n)}(\mathbf{k},\tau) = \mathcal{H}(\tau)a^n(\tau)\theta_n(\mathbf{k})$$

- $(a \propto \tau^2,\ \mathcal{H} = 2/\tau)$

Recursive solution

$$n\delta_n(\mathbf{k}) + \theta_n(\mathbf{k}) = A_n(\mathbf{k}), \quad 3\delta_n(\mathbf{k}) + (1+2n)\theta_n(\mathbf{k}) = B_n(\mathbf{k}),$$

where

$$A_n(\mathbf{k}) = -\int d^3q_1\, d^3q_2\, \delta^3(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{k}) \frac{\mathbf{k}\cdot\mathbf{q}_1}{q_1^2} \sum_{m=1}^{n-1} \theta_m(\mathbf{q}_1)\delta_{n-m}(\mathbf{q}_2),$$

$$B_n(\mathbf{k}) = -\int d^3q_1\, d^3q_2\, \delta^3(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{k}) \frac{k^2(\mathbf{q}_1\cdot\mathbf{q}_2)}{2q_1^2q_2^2} \times \sum_{m=1}^{n-1} \theta_m(\mathbf{q}_1)\theta_{n-m}(\mathbf{q}_2).$$

- Plug in to fluid equations: nth order term sourced by lower orders

Integral solution

- Can obtain explicit integral expression

$$\delta_n(\mathbf{k}) = \int d^3q_1 \dots d^3q_n \, \delta^3(\sum \mathbf{q}_i - \mathbf{k}) F_n(\{\mathbf{q}_i\}) \delta_1(\mathbf{q}_1) \dots \delta_1(\mathbf{q}_n)$$

$$\theta_n(\mathbf{k}) = \int d^3q_1 \dots d^3q_n \, \delta^3(\sum \mathbf{q}_i - \mathbf{k}) G_n(\{\mathbf{q}_i\}) \delta_1(\mathbf{q}_1) \dots \delta_1(\mathbf{q}_n)$$

- Kernels F_n, G_n defined recursively, first few are

$$F_1(\mathbf{q}_1) = G_1 = 1$$

$$F_2(\mathbf{q}_1, \mathbf{q}_2) = \frac{5}{7} + \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{2q_1q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{2}{7}\left(\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1q_2}\right)^2$$

$$G_2(\mathbf{q}_1, \mathbf{q}_2) = \frac{3}{7} + \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{2q_1q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{4}{7}\left(\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1q_2}\right)^2$$

Second order power spectrum

- Assume initial density δ_0 is a Gaussian random field, so all n-point functions reduce to products of 2-point function
- Expand δ to third order to obtain $P(k)$ to second order:

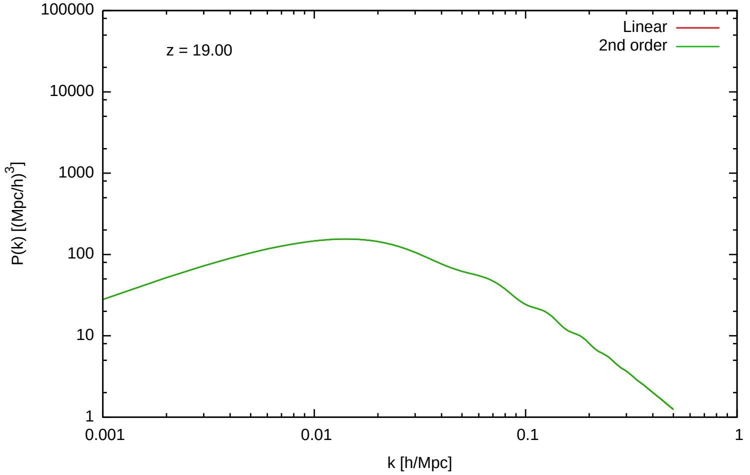
$$\begin{aligned}\langle\tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{k}')\rangle &= a^2\langle\tilde{\delta}_1(\mathbf{k})\tilde{\delta}_1(\mathbf{k}')\rangle + a^4\langle\tilde{\delta}_2(\mathbf{k})\tilde{\delta}_2(\mathbf{k}')\rangle \\ &\quad + a^4\langle\tilde{\delta}_1(\mathbf{k})\tilde{\delta}_3(\mathbf{k}')\rangle + a^4\langle\tilde{\delta}_3(\mathbf{k})\tilde{\delta}_1(\mathbf{k}')\rangle \\ \implies P_2(k) &= P_L(k) + P_{22}(k) + P_{13}(k)\end{aligned}$$

- Explicit integral expressions exist for P_{22} and P_{13}
- Schematically $P_{22} \sim \int P_L \int P_L, \quad P_{13} \sim P_L \int P_L$

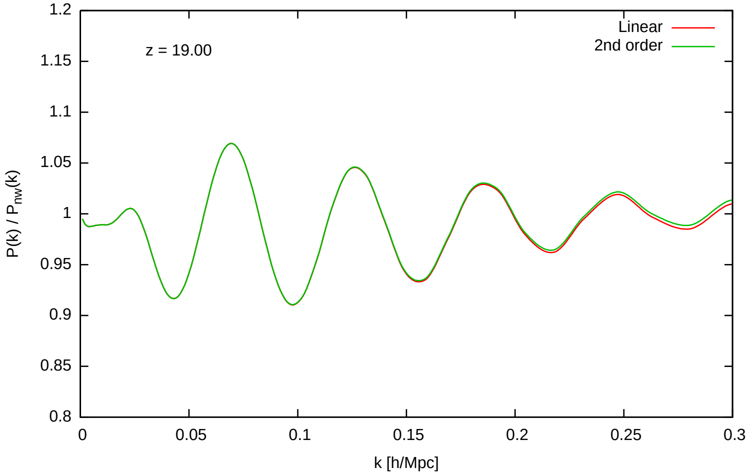
Limitations of SPT

- Only formally valid for Einstein-de Sitter universe: $D \propto a$
  - Approximately valid for arbitrary cosmology if we just replace a by the true linear growth function D in our perturbation expansion
- Perturbation theory breaks down at late times or at high k ($\sim k = 0.2h/\text{Mpc}$ at $z = 0$)
- Power spectrum diverges, can't calculate correlation function meaningfully

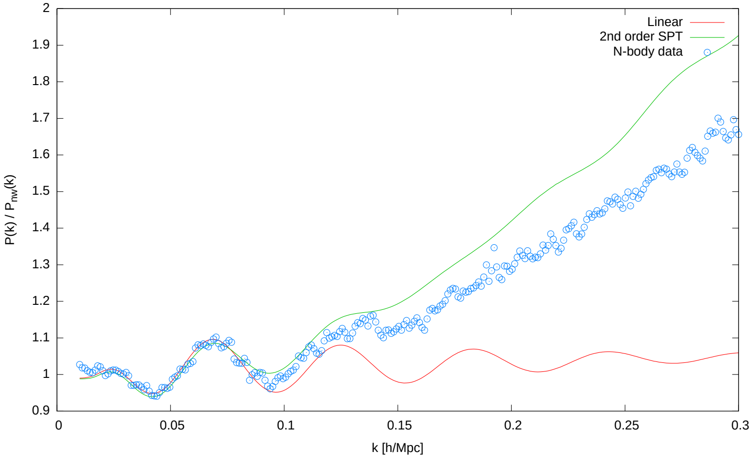
Growth of non-linear power



Growth of non-linear power



Comparison with N-body simulations



Lagrangian Perturbation Theory

- Lagrangian picture of fluid mechanics: $\mathbf{x} = \mathbf{q} + \mathbf{\Psi}(\mathbf{q})$
- $\bar{\rho}[1+\delta(\mathbf{x})]d^3x = \bar{\rho}d^3q \implies 1+\delta(\mathbf{x}) = [\det(\delta_{ij}+\partial\Psi_i/\partial q_j)]^{-1}$
- Linear solution for $\mathbf{\Psi}$ gives Zel'dovich approximation:

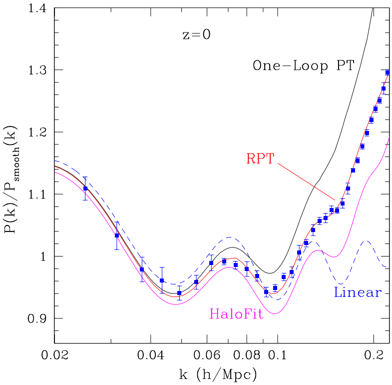
$$1 + \delta(\mathbf{x}, \tau) = \frac{1}{[1 - \lambda_1 D_1(\tau)][1 - \lambda_2 D_1(\tau)][1 - \lambda_3 D_1(\tau)]}$$

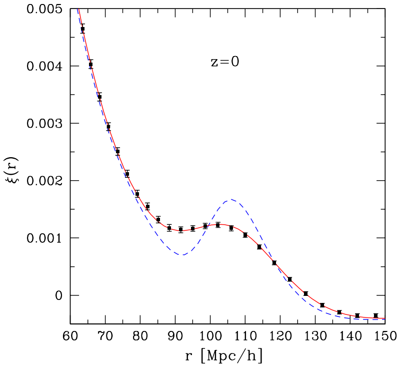
- Pros: intrinsically non-linear, 2nd and 3rd order calculations feasible
- Cons: breaks down at lower k than SPT

Renormalized Perturbation Theory

- Crocce & Scoccimarro, `astro-ph/0509418`
- Starts with diagrammatic formulation of perturbation expansion
- Attempts to identify renormalized vertices and propagators, *à la* QFT
- Pros: seems to match simulation data well
- Cons: extremely complicated, requires field theory background

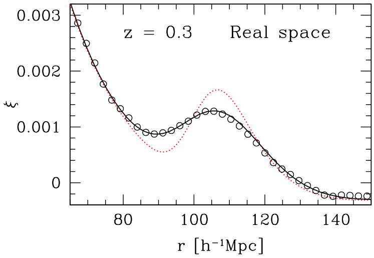
Renormalized Perturbation Theory





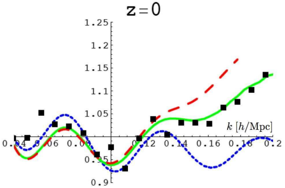
Resummation in Lagrangian picture

- Matsubara, `arXiv:0711.2521`
- Pulls out certain series of terms from infinite PT expansion, resums them into a Gaussian prefactor: $P \sim e^{-Ak^2}[P_L(k) + \tilde{P}_{13}(k) + P_{22}(k)]$
- Power spectrum is wrong at high k, but correlation function is good



Renormalization group techniques

- McDonald, astro-ph/0606028
- Macarrese and Pietroni, astro-ph/0703563
- Pietroni, arXiv:0806.0971



The future?

- Upcoming surveys need to be compared against accurate theoretical predictions to learn about dark energy
- Renewed interest in cosmological perturbation theory on many fronts
- Many new papers, with new techiques, appearing in recent years (even days!)

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